Assessing the quality of identified models through the asymptotic theory — when is the result reliable?☆

S. Garatti, M.C. Campi, S. Bittanti

Abstract

In this paper, the problem of estimating uncertainty regions for identified models is considered. A typical approach in this context is to resort to the asymptotic theory of Prediction Error Methods for system identification, by means of which ellipsoidal uncertainty regions can be constructed for the uncertain parameters.

We show that the uncertainty regions worked out through the asymptotic theory can be unreliable in certain situations, precisely characterized in the paper.

Then, we critically analyze the theoretical conditions for the validity of the asymptotic theory, and prove that the asymptotic theory also applies under new assumptions which are less restrictive than the usually required ones. Thanks to this result, we single out the classes of models among standard ones (ARX, ARMAX, Box–Jenkins, etc.) where the asymptotic theory can be safely used in practical applications to assess the quality of the identified model.

These results are of interest in many applications, including iterative controller design schemes.

Keywords: System identification; Model quality assessment; Asymptotic theory; ARMAX; Box–Jenkins

1. Introduction

Consider a data-generating dynamical system $P$ and a model $\hat{P}$ of it estimated from data. It has been fully recognized in the literature that the estimated model $\hat{P}$ is of little use without a statement on its quality. In other words, it is fundamental to characterize the error model, i.e. the distance between $P$ and $\hat{P}$ (see e.g. Bittanti & Picci, 1996; Goodwin, 1999; Kosut, Goodwin, Polis, 1992; Ljung, 1999b; Ninness & Goodwin, 1995; Söderström & Åström, 1995).

The most commonly used tool for evaluating the error model is the asymptotic theory of Prediction Error Methods (PEM) for system identification. It returns ellipsoidal confidence regions in the space of parameters such that the true system parameters belong to this ellipsoid with a specified probability (see e.g. Ljung, 1999a; Söderström & Stoica, 1989).

The main advantage of using the asymptotic theory is that the confidence regions can be easily computed from the available data. Moreover, these confidence regions are often reliable and give a tight description of uncertainty.

On the other hand, asymptotic theory has its own drawbacks too.

First, its applicability substantially requires the absence of un-modelled dynamics, while in real applications this assumption never applies (even if in many cases it does approximately). The importance of undermodelling is witnessed by many recent contributions such as Goodwin, Gevers, and Ninness (1992); Hakvoort and Van den Hof (1997); Hjalmarsson and Ljung (1992); Ljung and Guo (1997); Ljung (1999b, 2000, 2001); Reinelt, Garulli, and Ljung (2002) and Tjarnstrom and Ljung (1999). In order to overcome the problems encountered when the system

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Brett Ninness under the direction of Editor Torsten Söderström.

* Corresponding author. Marco C. Campi. Tel.: +39-030-3715458; fax: 39-030-380014.

E-mail addresses: sgaratti@elet.polimi.it (S. Garatti), campi@ing.unibs.it (M.C. Campi), bittanti@elet.polimi.it (S. Bittanti).
order is unknown, certain formulas valid for both the model order and the number of data points growing unbounded have been derived, see e.g. Ljung (1985, 1999a), Ninness, Hjalmarsson, and Gustaffson (1999) and Xie and Ljung (2001).

A second drawback is that the asymptotic theory is rigorously correct only when the number of data tends to infinity in such a way that the total amount of information on the system parameters grows unbounded. On the other hand, in real applications it often happens that the amount of excitation is substantial for certain parameters while there is a lack of information on other parameters (poor excitation). As a consequence, the asymptotic theory is used as a heuristic tool for the model quality evaluation.

In this paper, we focus attention on the problems arising when data are not informative enough, and one of our aims is to pinpoint the situations where the asymptotic theory may fail to provide sensible results with poor excitation. In these situations, the estimated parameters are subject to large uncertainty levels and the asymptotic theory can as well provide misleading results. This is quite a severe limitation since assessing the model quality is especially important for large uncertainty levels. Indeed, in the opposite case, the estimated model can be safely used in place of the true system with no particular need for an evaluation of its uncertainty. This leads to our first contribution:

(i) By way of an example, we explain why the asymptotic theory may fail for the model quality evaluation in presence of a high level of uncertainty.

We also note that this result is relevant to iterative control schemes where the closed-loop bandwidth is very restricted at the first iterations leading to poorly exciting signals and, in turn, to wide uncertainty in the estimated model (see Bittanti, Campi, Garatti, 2002; Gevers, 2000; Lee, Anderson, Kosut, & Mareels, 1993; Van den Hof & Schrama, 1995).

We next move to establish the situations where the asymptotic theory does not suffer from the problem highlighted in point (i) above, and it turns out that the asymptotic theory provides sensible results or not depending on the model class in which the data-generating system is identified. Our second contribution can be summarized as follows:

(ii) We single out the model classes among standard ones (ARX, ARMAX, Box–Jenkins, etc.) such that the asymptotic theory can be safely used to assess model quality, even in presence of a high level of uncertainty.

This latter result is made possible by a new asymptotic result, valid under relaxed assumptions, also worked out in this paper.

A different approach can be adopted in the analysis of uncertainty in the estimate by explicitly considering the finiteness of the data record. For some recent contributions along this line see Campi and Weyer (2002); Campi, Weyer, and Ooi (2002) and Weyer and Campi (2002).

1.1. Structure of the paper

In Section 2, our working assumptions are stated and a brief summary of the classical asymptotic theory is given. This allows us to keep the paper self-contained. Section 3 delivers the example as explained in point (i) above. After a mid-paper conclusion section (Section 4), Section 5 contains the new asymptotic result valid under relaxed assumptions. In Section 6, we move to consider the quality assessment with finite data points and show the relevance of the theorem in Section 5 to this purpose. Finally, in Section 7 the classes of models to which the asymptotic theory can be safely applied for model quality estimation are singled out, while some illustrative simulations are given in Section 8.

2. Asymptotic theory of PEM

In this section we provide a compendium of the asymptotic theory of PEM for system identification with the objective of clarifying the context of our results. For a more comprehensive description of the subject, we refer the reader to the literature (see e.g. Ljung, 1999a; Söderström & Stoica, 1989).

2.1. Mathematical setting

Let

\[ M_0 = \{ \hat{y}(t, \vartheta) = W_d(z^{-1}, \vartheta)u(t) + W_s(z^{-1}, \vartheta)y(t), \vartheta \in \Theta \subseteq \mathbb{R}^n \} \]

be a parameterized set of predictor models, where \( W_d(z^{-1}, \vartheta) \) and \( W_s(z^{-1}, \vartheta) \) satisfy the following assumption.

Assumption 1. \( W_d(z^{-1}, \vartheta) \) and \( W_s(z^{-1}, \vartheta) \) are rational strictly proper (as functions in \( z \)) transfer functions whose coefficients are functions of a parameter \( \vartheta \in \Theta \), where \( \Theta \) is a nonempty compact set in \( \mathbb{R}^n \). The coefficients are four times differentiable with respect to \( \vartheta \) and the fourth derivatives are continuous. Moreover, \( W_d(z^{-1}, \vartheta) \) and \( W_s(z^{-1}, \vartheta) \) are asymptotically stable, \( \forall \vartheta \in \Theta \).

Remark 1. In the classical asymptotic theory, the coefficients of the transfer functions \( W_d(z^{-1}, \vartheta) \) and \( W_s(z^{-1}, \vartheta) \) are usually only required to be twice differentiable with continuous second derivatives. Here, the assumption has been strengthened in view of our further results. It is perhaps worth mentioning that for standard identification model classes (ARX, ARMAX, Box–Jenkins, etc.) the coefficients are the parameters themselves, so that the differentiability assumption is not an issue.

\( u \) and \( y \) are respectively the input and output of the system, and are generated according to the following scheme.
Assumption 2. Processes $u$ and $y$ are given by
\[ u(t) = G_u(z^{-1})r(t) + H_u(z^{-1})e(t), \]
\[ y(t) = G_y(z^{-1})r(t) + H_y(z^{-1})e(t), \]
where $G_u(z^{-1})$, $G_y(z^{-1})$, $H_u(z^{-1})$ and $H_y(z^{-1})$ are asymptotically stable rational transfer functions. $e(t)$ is a zero mean independent process with constant variance equal to $\delta^2 > 0$ and such that $\sup E[|e(t)|^{4+\delta}] < \infty$, for some $\delta > 0$. $r(t)$ is a wide sense stationary, ergodic, stochastic, external input sequence. $e(t)$ and $r(t)$ are independent.

Remark 2. The results given below can be proved even if $r(t)$ is a bounded deterministic external input sequence. Considering a stationary, ergodic, recursive identification, in Assumption 2 has been preferred since it simplifies the presentation.

Remark 3. Note that Assumption 2 encompasses closed-loop as well as open-loop configurations. In the latter, $H_u(z^{-1}) = 0$ and $G_u(z^{-1}) = 1$.

We also require that the data-generating system belongs to the class of models $\mathcal{M}_\theta$, that is:

Assumption 3. There exists a parameter $\theta^0$, which is an interior point of $\Theta$, such that
\[ y(t) = W_u(z^{-1}, \theta^0)u(t) + W_y(z^{-1}, \theta^0)y(t) + e(t). \]

Remark 4. When the data-generating system does not belong to the assumed class of model $\mathcal{M}_\theta$, the system-model mismatch comprises two terms: a variance term and a bias term. In this case the asymptotic theory applies so as to only assess the variance term at the price of a more complicated formulation that accounts for the correlation in the residue due to the bias term. See e.g., Hakvoort and Van den Hof (1997) and Hjalmarsson and Ljung (1992).

Parameter $\theta$ is estimated by the minimization of the standard quadratic cost:
\[ V_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} \varphi(t_i, \theta)^2, \]
where $N$ is the number of data points and $\varphi(t_i, \theta) = y(t) - \hat{y}(t_i, \theta)$ is the prediction error.

Thus, the estimate is
\[ \hat{\theta}_N = \arg \min_{\theta} V_N(\theta). \]

The asymptotic cost criterion is $V(\theta) = \mathbb{E}[\varphi(t, \theta)^2]$, and we will denote by $\Theta^*$ the corresponding set of minimizers within the feasible set $\Theta$, that is
\[ \Theta^* = \{ \arg \min_{\theta \in \Theta} V(\theta) \}. \]

In the classical asymptotic theory it is assumed that $V(\theta)$ has a unique minimizer:

Assumption 4. The set $\Theta^*$ has cardinality equal to 1.

Remark 5. Under Assumption 3, it is easy to demonstrate that the parameter $\theta^0$ always belongs to the set $\Theta^*$. Therefore, Assumption 4 can be rewritten as $\Theta^* = \{ \theta^0 \}$. 2.2. Asymptotic theory results

Let
\[ Q_N = \frac{(1/N) \sum_{i=1}^{N} \psi_i(t_i, \hat{\theta}_N(t_i, \hat{\theta}_N)^2)}{(1/N) \sum_{i=1}^{N} \varphi_i(t_i, \hat{\theta}_N)^2}, \]
where $\psi_i(t_i, \theta)$ denote $(\partial/\partial \theta) \varphi_i(t, \theta)$, and consider the following ellipsoid centered in $\hat{\theta}_N$:
\[ \delta(r) = \{ \theta : (\hat{\theta}_N - \theta)^T Q_N (\hat{\theta}_N - \theta) \leq r \}, \]
where $r$ is a real positive number called the size of the ellipsoid.

The standard result of the asymptotic theory writes as follows:

Theorem 1. Let $p \in [0, 1)$ and assume that $(\partial^2/\partial \theta^2) \tilde{V}(\theta_0) > 0$. Under Assumptions 1, 2, 3 and 4, it follows that:
\[ \lim_{N \to \infty} \mathbb{P} \left\{ \tilde{V}(\theta) \leq \frac{\alpha(p)}{N} \right\} = p, \]
where $\alpha(p)$ is the inverse of the function $p = \int_0^\infty f_{\tilde{V}}(x) \, dx$ and $f_{\tilde{V}}(x)$ is the probability density of a $\chi^2$ random variable with $n$ degrees of freedom.

The above theorem suggests how to select $r$ so as to obtain an ellipsoid confidence region for $\theta_0$ of pre-assigned asymptotic probability $p$. The proof of Theorem 1 can be found in Chapter 9 of Ljung (1999a).

The following result is obtained immediately from Theorem 1.

Theorem 2. Assume that $(\partial^2/\partial \theta^2) \tilde{V}(\theta_0) > 0$. Under Assumptions 1, 2, 3 and 4, for any sequence $\alpha_N$ which tends to $\infty$ as $N \to \infty$, we have that
\[ \lim_{N \to \infty} \mathbb{P} \left\{ \tilde{V}(\theta_0) < \frac{\alpha_N}{N} \right\} = 1. \]

Remark 6. As a natural choice for $\alpha_N$, consider $\alpha_N = \alpha(p)(1 + \beta_N)$, for some $p$, with $\beta_N \to \infty$ as $N \to \infty$, that is, the ellipsoid size is inflated by the factor $1 + \beta_N$ with respect to Theorem 1. If $\beta_N/N \to 0$, when $N \to \infty$, the ellipsoid size still tends to zero, though with a slower rate than the ellipsoid of Theorem 1. Theorem 2 says that, no matter how slow such an inflation takes place, the true parameter $\theta_0$ will asymptotically belong to the ellipsoid with confidence 1. A good choice of $\beta_N$ is reliant on the specific problem at hand and its value is dictated by experience.

In real applications, the asymptotic theory is often used to generate confidence regions for the system parameters, even
if, as is obvious, such a theory applies only approximately since the evaluation is based on a finite number of data points. Though it is common experience that the results are still reliable in many cases even for a moderate data sample, it is also true that in other situations the asymptotic theory may fail to provide sensible indications, even for a large set of data points.

The goal of the present paper is to give a clearcut view of the situations in which this actually occurs and to pinpoint the model classes for which the asymptotic theory can be safely used. We start in the next section with an example clarifying where the trouble may come from in the use of the asymptotic theory.

3. An example where the asymptotic theory provides misleading results with poorly informative data

Consider the following data-generating system:

\[ y(t) = \frac{b^0 z^{-1}}{1 + d^0 z^{-1}} u(t) + (1 + h^0 z^{-1}) e(t), \]

where \( d^0 = -0.7, b^0 = 0.3, h^0 = 0.5 \) and \( e(t) \sim WGN(0, 1) \) (\( WGN = \) White Gaussian Noise). In addition, the plant is operated in closed loop as shown in Fig. 1. It is a trivial task to verify that the closed loop system is stable. \( N = 10,000 \) data points \((u, y)\) have been collected when the system was operated with a reference signal \( r(t) \sim WGN(0, 1) \), independent of \( e(t) \) (note that the variance of the reference signal is very small as compared to the noise variance—poor excitation).

Based on the \((u, y)\) measurements, a full-order model for the data-generating system (6) has been identified and a confidence region \( \mathcal{E}(\hat{\pi}(p)/N) \), \( p = 0.99 \), has also been estimated through the asymptotic Theorem 1.

The amplitude Bode diagrams of the identified model and of the real system \( u \) to \( y \) transfer functions have been plotted in Fig. 2.

From the plot, a wide mismatch between the real plant and the identified model is apparent. This is not surprising, since the reference signal is poorly exciting. On the other hand, we would also expect that the uncertainty region supplied by the asymptotic theory is wide.

Fig. 3 displays the confidence region \( \mathcal{E}(\hat{\pi}(p)/N) \) in the frequency domain. Surprisingly, the confidence region concentrates around the identified model, showing that the model quality assessment is completely unreliable in this case.

It is perhaps interesting to note that the presented situation—though admittedly artificial—is a simplification of what often happens in practical identification, where poor excitation is due to a restricted bandwidth of the closed-loop system. The simplified situation of a poorly exciting external signal \( r(t) \) has been adopted here for ease of presentation.

3.1. Explanation

Let us briefly explain the mechanism that made the model quality estimation unreliable in the present situation.

The explanation becomes easier if we assume that the reference signal is exactly equal to zero. For this reason we
It is important to reassert the fact that such a behavior is a consequence of the poorness of the available information. In turn, this is primarily due to the poor excitation conveyed by each single data point (since \( r(t) \) is very small) and secondarily to the finiteness of the number of data points (so that the total amount of information in the data is limited).

In order to explain why the confidence region provided by the asymptotic theory is not reliable, it is, at this point, necessary to recall an aspect of the asymptotic theory which is relevant to the present discussion (see Ljung, 1999a; Söderström & Stoica, 1989 for details).

Theorems 1 and 2 are both based on the following fundamental expansion:

\[
0 = \sqrt{N} \frac{d}{d\theta} V_N(\hat{\theta}_N) = \sqrt{N} \frac{d}{d\theta} V_N(\theta^0) + \frac{d^2}{d\theta^2} V_N(\xi_N) \sqrt{N}(\hat{\theta}_N - \theta^0). \tag{8}
\]

This equation is nothing but the Taylor expansion of \( (d/d\theta)V_N \) (where all terms are inflated by the coefficient \( \sqrt{N} \) and \( \xi_N \) is a point between \( \theta^0 \) and \( \hat{\theta}_N \)). The evaluation of the confidence region for \( \hat{\theta}_N - \theta^0 \) is carried out by observing that: first, \( \sqrt{N}(d/d\theta)V_N(\theta^0) \) is asymptotically a zero mean Gaussian random variable; second, \((d^2/d\theta^2)V_N(\xi_N)\) converges to \((d^2/d\theta^2)\hat{\gamma}(\theta^0)\), since \( \hat{\theta}_N \to \theta^0 \) so that \( \xi_N \) is squeezed towards \( \theta^0 \). The quantity \((d^2/d\theta^2)\hat{\gamma}(\theta^0)\) is further approximated by \((d^2/d\theta^2)\gamma_N(\hat{\theta}_N)\) leading to the asymptotic Theorems 1 and 2.

If \( \hat{\theta}_N \) is sufficiently close to \( \theta^0 \), this last approximation concerning the second derivative has a negligible effect. However, in the previous example this is not so, since the estimate \( \hat{\theta}_N \) is trapped far from \( \theta^0 \) and this is the reason for the misleading result as shown in Fig. 3.

Let us explain more in detail the mechanism through which such a misleading result is generated.

Due to the effect of the inflating coefficient \( \sqrt{N} \), \( \sqrt{N}(\hat{\theta}_N - \theta^0) \) takes on quite a large value. Despite this, Eq. (8) holds true (Eq. (8) is always true since it contains no approximation). In fact, in (8) \((d^2/d\theta^2)\gamma_N(\xi_N)\) is computed in a point \( \xi_N \) between \( \theta^0 \) and \( \hat{\theta}_N \) where \((d^2/d\theta^2)\gamma_N(\xi_N)\) is almost singular, leading to a term \((d^2/d\theta^2)\gamma_N(\xi_N)\sqrt{N}(\hat{\theta}_N - \theta^0)\) of moderate magnitude. Unfortunately, as explained before, \( \xi_N \) is not accessible and \((d^2/d\theta^2)\gamma_N(\xi_N)\) is substituted by \((d^2/d\theta^2)\gamma_N(\hat{\theta}_N)\) which turns out to be well positive definite. This leads to the mistaken conclusion that \( \hat{\theta}_N - \theta^0 \) is small and to the unreliable uncertainty region shown in Fig. 3.

Note that a second interpretation of the obtained result is also possible: For \( r(t) = 0 \) the found region is in fact a confidence region around \( \theta^0 \), the spurious minimizer different from \( \theta^0 \). When \( r(t) = WGN(0, 10^{-6}) \) the found confidence region can be interpreted as a perturbation of the previous one.
4. Mid-paper conclusions

The results of the previous sections can be summarized as follows:

(i) The classical asymptotic theory requires that the asymptotic cost criterion has a unique minimizer \( \hat{\theta}^* = \theta^0 \); moreover if data are poorly informative so that \( \hat{\theta}_N \) is not close enough to \( \theta^* \) (wide uncertainty), then the resulting uncertainty evaluation by means of \( \epsilon(\varepsilon(p)/N) \) can be unreliable, i.e. the asymptotic theory results do not hold, even approximately.

(ii) Due to (i), a blind application of the asymptotic theory can lead to misleading results.

In the next sections our goal is to study the situations where the asymptotic theory provides reliable results, even when \( \hat{\theta}_N \) is far from \( \theta^0 \). To this purpose, we proceed along the following lines:

(iii) We extend the asymptotic theory results so as to encompass the case of multiple minimizers of the asymptotic cost criterion \( \hat{\tilde{V}}(\theta) \) (Section 5).

(iv) Thanks to the result of point (iii), we show that— if a suitable additional condition on the model class is satisfied—then the asymptotic theory can be safely used even for a high level of uncertainty, namely for \( \hat{\theta}_N \) far from \( \theta^0 \) (Section 6).

(v) We establish which standard model classes (ARMAX, Box–Jenkins, etc.) satisfy the additional condition of point (iv) (Section 7).

5. A new asymptotic result

In this section, we provide a new asymptotic result which generalizes the standard asymptotic theory of Section 2. The fact that this result is useful when data are poorly informative is discussed in Section 6.

Assumption 4 in Section 2 is here replaced by the following one.

**Assumption 4’**. \( \Theta^* = \mathcal{S} \cap \Theta \), where \( \mathcal{S} \) is an affine subspace of the parameter space \( \mathbb{R}^n \).

Moreover, \( \hat{\theta}_N \rightarrow \theta^* \) (not necessary equal to \( \theta^0 \)) almost surely, where \( \theta^* \in \Theta^* \) is an interior point of \( \Theta \).

**Remark 7**. Note that Assumptions 1, 2, 3 and 4’ are more general than Assumptions 1, 2, 3 and 4. Indeed, Assumption 4 implies that \( \Theta^* = \{ \theta^0 \} \), so that the first part of Assumption 4’ holds with \( \mathcal{S} = \{ \theta^0 \} \), which is an affine subspace (it is the origin of \( \mathbb{R}^n \) translated).

As for the second part of Assumption 4’, it holds under Assumptions 1, 2, 3 and 4 with \( \theta^* = \theta^0 \).

**Remark 8**. In Assumption 4 the important fact is that \( \Theta^* = \text{linearly structured} \) (apart from the fact that it is confined to \( \Theta \)).

In the following Theorem 3 we show that the asymptotic Theorem 2 can be preserved in the present setting.

**Theorem 3**. Assume that \((d^2/d\theta^2)\hat{\tilde{V}}(\theta^*)\) is positive definite along the directions of \( \mathcal{S} \perp \) (the subspace orthogonal to \( \mathcal{S} \)).

Under Assumptions 1, 2, 3 and 4’, for any sequence \( \varepsilon_N \) which tends to \( \infty \) as \( N \rightarrow \infty \), we have that (see (5) for the definition of \( \epsilon(\varepsilon) \)):

\[
\lim_{N \rightarrow \infty} P\left\{ \theta^0 \in \epsilon\left( \frac{\varepsilon_N}{N} \right) \right\} = 1.
\]

**Proof**. See the Appendix.

**Remark 9**. In contrast to Theorems 1 and 2, in Theorem 3 the positive definiteness of \((d^2/d\theta^2)\hat{\tilde{V}}(\theta^*)\) is only required in the directions of \( \mathcal{S} \perp \). In this connection, one could note that \((d^2/d\theta^2)\hat{\tilde{V}}(\theta^*)\) is in fact singular in the direction of \( \mathcal{S} \) due to Assumption 4’.

**Remark 10**. Allowing for multiple minimizers of \( \hat{\tilde{V}}(\theta) \), as is done in Theorem 3, permits to cope with situations where there is a lack of excitation (see Section 6 for further discussion).

**Remark 11**. Similarly to Remark 4 we mention here that the results of the new asymptotic Theorem 3 can be extended to the case in which the model class does not contain the true system. Clearly, in full analogy with Remark 4 this allows one to assess the variance term only, so that the ensuing results are perhaps less interesting than in the full order case. Details are omitted as a complete discussion of the matter would lead us too far afield.

6. Use of Theorem 3 in practice

As we have seen in Section 3, in certain cases applying the asymptotic formulas to assess the quality of the identified model can lead to misleading conclusions.

Here, we want to show that, under an additional condition on the model class, the asymptotic formulas can indeed be safely used for such an evaluation. This conclusion is possible in the light of the new asymptotic result stated in the previous section.

Let us go back for a moment to the example of Section 3. There, if \( r(t) = 0 \), then \( \hat{\tilde{V}}(\theta) \) has two global isolated minimizers. When we performed the identification of the plant, instead of minimizing \( \hat{\tilde{V}}(\theta) \) we had of course to resort to its empirical counterpart \( V_N(\theta) \); moreover, \( r(t) \) was small, but not equal to 0. Thus, the actual identification optimization setting can be seen as a perturbed setting with respect to the ideal one where one minimizes \( \hat{\tilde{V}}(\theta) \) with \( r(t) = 0 \).
As we have seen, $\hat{\theta}_N$ can possibly fall near the minimizer of the ideal setting which does not correspond to the true system. If so, the asymptotic formulas lead to computing a deceivingly small uncertainty region.

We now introduce the following additional condition on the model class:

**Condition 1.** Independently of the level of excitation in the signals, the set of the minimizers of $\hat{V}(\vartheta)$ is an affine subspace.

**Remark 12.** Condition 1, as is obvious, can be rewritten as: For every excitation level of the signals, there exists an affine subspace $\mathcal{S}$ such that $\Theta^* = \mathcal{S} \cap \Theta$. However, the reader should note that this requirement is different from the first part of Assumption 4 where one requires that $\Theta^* = \mathcal{S} \cap \Theta$ only for $\Theta^*$ arising in the particular operating condition, i.e. for a fixed level of excitation of the input signal.

Now, suppose that a model class fulfilling Condition 1 is used. If we are in an ideal situation with a complete lack of excitation, then $\hat{V}(\vartheta)$ is minimized in an affine subspace, say $\mathcal{S}$, and Theorem 3 can be applied to this situation. If instead we are in a real identification setting where we minimize $V_N(\vartheta)$ and, possibly, some extra degree of excitation is added to the signals, this real setting can be seen as a perturbed setting of the ideal one. Thus, though $\hat{\theta}_N$ is far from $\vartheta^0$, Theorem 3 still holds approximately and formula (9) can be used for the model quality assessment.

As it appears, the asymptotic theory can be safely applied with poorly exciting data to the model classes for which the set of minimizers of $\hat{V}(\vartheta)$ is an affine subspace. Studying these classes is the subject of the next section, while simulation examples illustrating the result are shown in Section 8.

7. Assessment of the model classes for which $\hat{V}(\vartheta)$ is minimized in an affine subspace

We treat separately two different situations, namely open-loop identification and closed-loop identification as these two settings give different results.

7.1. Open-loop identification

By “open-loop identification” we mean that the input signal $u(t)$ and the noise signal $e(t)$ are independent. Technically speaking, this is equivalent to taking $H_u(z^{-1}) = 0$ and $G_n(z^{-1}) = 1$ in Assumption 2.

**Theorem 4.** Let $\mathcal{M}_\vartheta$ be the Box–Jenkins (BJ) class of predictor models, i.e.

$$\mathcal{M}_\vartheta = \{ \hat{y}(t, \vartheta) = (1 - H(z^{-1}, \vartheta)z^{-1})y(t) + H(z^{-1}, \vartheta)z^{-1}G(z^{-1}, \vartheta)u(t), \vartheta \in \Theta \},$$

where $G$ and $H$ are rational transfer functions, $H(0, \vartheta) = 1$, $\forall \vartheta \in \Theta$, and $\vartheta$ is a vector containing the numerator and denominator polynomial coefficients of $G$ and $H$.

Suppose that the identification is performed in open-loop and that Assumptions 1, 2 and 3 are satisfied.

Then, Condition 1 holds true.

**Proof.** See the Appendix. $\square$

Theorem 4 can be applied to Output Error (OE) models as well, since OE is a particular case of BJ. In fact, we remind the reader that this requirement is different from the OE predictor model class

$$\mathcal{M}_\vartheta = \{ \hat{y}(t, \vartheta) = G(z^{-1}, \vartheta)u(t), \vartheta \in \Theta \},$$

where $G$ is a rational transfer function and $\vartheta$ is the vector of the numerator and denominator polynomial coefficients of $G$.

Even though Theorem 4 does not apply directly, a result similar to Theorem 4 holds for ARX and ARMAX models too. In this case,

$$\mathcal{M}_\vartheta = \left\{ \hat{y}(t, \vartheta) = \left( 1 - \frac{A(z^{-1}, \vartheta)}{C(z^{-1}, \vartheta)} \right) y(t) \right\}$$

$$+ \frac{B(z^{-1}, \vartheta)}{C(z^{-1}, \vartheta)} u(t), \vartheta \in \Theta \},$$

where $A, B$ and $C$ are polynomials in $z^{-1}, A$ and $C$ are monic, and $\vartheta$ is the vector of the coefficients of these polynomials (the ARX case corresponds to $C(z^{-1}, \vartheta) = 1$). One should note that in the ARX and ARMAX structures, $G(z^{-1}, \vartheta)$ and $H(z^{-1}, \vartheta)$ are not freely parameterized as assumed in Theorem 4. However, the proof of this theorem can be extended with minor amendments to cover the ARX and ARMAX cases.

It is perhaps worth mentioning that not all model structures satisfy Condition 1 even in open-loop. An example is given by the model class

$$A(z^{-1}, \vartheta)y(t) = G(z^{-1}, \vartheta)u(t) + H(z^{-1}, \vartheta)e(t)$$

which corresponds to the predictor model class

$$\mathcal{M}_\vartheta = \{ \hat{y}(t, \vartheta) = (1 - A(z^{-1}, \vartheta)H(z^{-1}, \vartheta)^{-1})y(t) + H(z^{-1}, \vartheta)G(z^{-1}, \vartheta)u(t), \vartheta \in \Theta \},$$

where $A$ is a monic polynomial in $z^{-1}, G$ and $H$ are rational transfer functions, $H(0, \vartheta) = 1, \forall \vartheta \in \Theta$, and $\vartheta$ is the vector of the coefficients of $A$ and of the numerator and denominator polynomial coefficients of $G$ and $H$. In Section 8, a simulation example involving this class of models is presented.

7.2. Closed-loop identification

Suppose now that the system is operated in closed-loop with a controller $R$ as in Fig. 5.

**Theorem 5.** Suppose that the identification is performed in closed-loop and that Assumptions 1, 2 and 3 are satisfied.
Then, Condition 1 holds true for the ARMAX and OE classes of models.

Proof. See the Appendix.

It has to be noted that, when identification is performed in closed-loop, the Box–Jenkins structure does not meet Condition 1 in general. In fact, the example presented in Section 3 was based on a Box–Jenkins model.

8. Simulation examples

8.1. Example—BJ model of Section 3 in open-loop

Consider again the data-generating system described in (6), but suppose now that the system is operated in open-loop with an input signal \( u(t) \sim WGN(0, 10^{-6}) \), independent of \( e(t) \). A full order model has been identified by means of the BJ model class with \( N = 10,000 \). An ellipsoidal confidence region \( \delta(\alpha_{N}/N) \), \( \alpha_{N} = \alpha(p) \), \( p = 0.99 \), has been also estimated.

The identified model is shown in Fig. 6. Again, as in Section 3, the model presents a large mismatch with the true system since the noise-to-signal ratio is large. Fig. 7 displays \( \delta(\alpha_{N}/N) \) in the frequency domain.

The uncertainty region is very scattered in this case, and covers the gap between the identified model and the true plant. Thus, the estimated uncertainty is reliable, in agreement with Theorem 4.

8.2. Example—a model class which does not meet Condition 1 in open-loop

Consider now the following data-generating system:

\[
(1 + a^0 z^{-1}) y(t) = b^0 z^{-1} u(t) + \frac{1}{1 + h^0 z^{-1}} e(t),
\]

where \( a^0 = -0.7 \), \( b^0 = 0.3 \), \( h^0 = 0.5 \) and \( e(t) \sim WGN(0, 1) \).

We have identified a full order model when the plant is operated in open-loop with a constant (poorly exciting) input signal \( u(t) = 1 \), \( \forall t \), and \( N = 10,000 \). An ellipsoidal confidence region \( \delta(\alpha_{N}/N) \), \( \alpha_{N} = \alpha(p) \), \( p = 0.99 \), has been also estimated.

System (11) belongs to the model class (10) and falls outside the realm of applicability of Theorem 4. The computed uncertainty region is displayed in Fig. 8, showing that the asymptotic theory provides unreliable results. As a matter of fact it is not difficult to see that Condition 1 is violated in this case. Indeed, a simple computation shows that:

\[
\bar{V}(\vartheta) = |1 + h|^2 \left| \frac{b^0}{1 + a^0} - \frac{b}{1 + a} \right|^2 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 + h z^{-1} \right| \left| 1 + a^0 z^{-1} \right|^2 |z = e^{i\omega}| d\omega,
\]

where \( \vartheta = [a \ b \ h]^T \).
The minimal value of $\bar{V}(\vartheta)$ is achieved only in the following two points:
\begin{align*}
b_1^* &= 1 + a_1^* \quad b_2^* = 1 + a_2^* \\
a_1^* &= a_0 \quad a_2^* = h_0 \\
h_1^* &= h_0 \quad h_2^* = a_0
\end{align*}
and, therefore, Condition 1 does not hold.

\section{Concluding remarks}

In this paper we have considered the problem of assessing the quality of identified models in a “Prediction Error” framework. Two main facts have been pointed out:

- In case of large uncertainty, the confidence regions supplied by the asymptotic theory may be unreliable;
- in spite of the presence of large uncertainty, the same confidence regions can be safely used if an extra condition holds true for the model class used in the identification procedure.

Moreover, we have provided a classification of the standard model classes (ARX, ARMAX, Box–Jenkins, etc.) that play an important role in a number of applications.

\section{Acknowledgements}

Paper supported by MIUR under the National Research Project “New methods for Identification and Adaptive Control for Industrial Systems”.

\section{Appendix A. Proofs of the results}

\subsection{A.1. Proof of Theorem 3}

We need a preliminary result.

\textbf{Lemma 1.} Let $\tilde{\vartheta}$ be a minimizer of $\bar{V}(\vartheta)$. Then, under Assumptions 1, 2, 3, it holds that
\begin{equation*}
\varrho(t, \tilde{\vartheta}) = e(t) \text{ almost surely.}
\end{equation*}

\textbf{Proof.} Since $\tilde{y}(t, \vartheta)$ depends on data up to time $t - 1$ only (see Assumption 1), predictor $\tilde{y}(t, \vartheta)$ and $e(t)$ are independent for any $\vartheta$.

Therefore, thanks to Assumption 3, we obtain that
\begin{align*}
\bar{V}(\vartheta) &= \mathbb{E}[(e(t) + \tilde{y}(t, \vartheta^0) - \tilde{y}(t, \vartheta))^2] \\
&= \mathbb{E}[e(t)^2] + \mathbb{E}[(\tilde{y}(t, \vartheta^0) - \tilde{y}(t, \vartheta))^2].
\end{align*}
Since $\tilde{\vartheta}$ minimizes $\bar{V}(\vartheta)$, the term $\mathbb{E}[(\tilde{y}(t, \vartheta^0) - \tilde{y}(t, \vartheta))^2]$ must be equal to 0 and this implies that $\tilde{y}(t, \vartheta^0) - \tilde{y}(t, \vartheta) = 0$ almost surely.

Finally, $\varrho(t, \tilde{\vartheta}) = y(t) - \tilde{y}(t, \vartheta) = y(t) - \tilde{y}(t, \vartheta^0) = e(t)$ almost surely. \hfill \square

\textbf{Proof of Theorem 3.} Recall that, by definition (5) of $\mathcal{E}(\cdot)$, the condition
\begin{equation*}
\vartheta^0 \in \mathcal{E}\left(\frac{2\sigma}{N}\right)
\end{equation*}
is equivalent to
\begin{equation*}
(\widehat{\vartheta}_N - \vartheta^0)^\top Q_N(\widehat{\vartheta}_N - \vartheta^0) \leq \frac{2\sigma}{N}.
\end{equation*}

As a consequence, the theorem can be proven by showing that
\begin{equation*}
\lim_{N \to \infty} \frac{N}{2\sigma} \cdot (\widehat{\vartheta}_N - \vartheta^0)^\top Q_N(\widehat{\vartheta}_N - \vartheta^0) = 0 \text{ in probability.}
\end{equation*}

(12)

Let $d$ be the dimension of the affine subspace $\mathcal{S}$. Then, let $x \in \mathbb{R}^d$ be the first $d$ (the remaining $n - d$) coordinates of $\vartheta$, that is $\vartheta = (x' z')'$. Thus, $\vartheta^0 = (x^0' z^0')'$ and $\widehat{\vartheta}_N = (\widehat{x}_N' \widehat{z}_N')'$.

Without loss of generality we assume that $\mathcal{S}$ is parallel to the hyperplane determined by the $x$ coordinates (this can be always achieved by a rotation of the axes). See Fig. 9 for a graphical representation of the parameter space when $\Theta \subset \mathbb{R}^2$ and $\mathcal{S}$ is a straight line.

We now prove Eq. (12).

In order to avoid notational cluttering, throughout we omit the $t$-dependence, e.g. $\psi(\widehat{\vartheta}_N)$ stands for $\psi(t, \widehat{\vartheta}_N)$. Moreover, $\sum$ is used for $\sum_{t=1}$. Fig. 8. Uncertainty region of the estimated model.
Since
\[ Q_N = \left( \frac{1}{N} \right) \sum \psi(\hat{\theta}_N) \psi(\hat{\theta}_N) \]
\[ \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
\[ \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
\[ \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
where \( e_x \) and \( e_z \) denote the vector of derivatives of \( \epsilon \) with respect to \( x \) and \( z \) coordinates, we have that
\[ \frac{N}{\sigma^2} (\hat{\theta}_N - \theta^0) Q_N (\hat{\theta}_N - \theta^0) \]
\[ = \frac{N}{\sigma^2} \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
\[ + 2 (\hat{\epsilon}_N - x^0)^T e_x (\hat{\theta}_N) \]
\[ + (\hat{\epsilon}_N - z^0)^T e_z (\hat{\theta}_N) \]
\[ \leq \frac{N}{\sigma^2} \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
\[ + 2 (\hat{\epsilon}_N - x^0)^T e_x (\hat{\theta}_N) \]
\[ + 2 (\hat{\epsilon}_N - z^0)^T e_z (\hat{\theta}_N) \]
\[ = \frac{N}{\sigma^2} \left( \frac{1}{N} \sum e(\hat{\theta}_N) \right)^2 \]
\[ \times \left( (\hat{\epsilon}_N - x^0)^T \sum e_x (\hat{\theta}_N) e_x (\hat{\theta}_N) \right) \]
\[ + (\hat{\epsilon}_N - z^0)^T \sum e_z (\hat{\theta}_N) e_z (\hat{\theta}_N) \)
\[ \to 0 \text{ in probability}, \quad \quad \quad \quad (13) \]

Let us first prove Eq. (14).

We first consider the term \( (1/N) \sum e_x (\hat{\theta}_N) e_x (\hat{\theta}_N) \) and prove that
\[ \frac{1}{N} \sum e_x (\hat{\theta}_N) e_x (\hat{\theta}_N) \to \frac{\hat{\phi}(x^*, z^*)}{2} \quad \text{almost surely.} \quad \quad \quad \quad (15) \]

Note that \( (1/N) \sum e_x (\hat{\theta}_N) e_x (\hat{\theta}_N) \to \mathbb{E}[e_x (x^*, z^*)] \) almost surely, as it follows from Assumptions 1, 2, 3 and 4 (in fact, this result is a consequence of the uniform convergence of empirical means for linear predictors — see Ljung (1978) and Theorem 2B.1 in Ljung (1999a)).

Thus, all we need to prove is
\[ \frac{1}{N} \sum e_x (\hat{\theta}_N) e_x (\hat{\theta}_N) \to \frac{\hat{\phi}(x^*, z^*)}{2}. \quad \quad \quad \quad (16) \]

We have that
\[ \left( \frac{\partial^2}{\partial \theta^2} \hat{\phi}(\theta^*) \right) = \mathbb{E} \left[ \frac{\partial^2}{\partial \theta^2} \psi(\theta^*) \right]. \]

Thus, \( (\partial^2/\partial \theta^2) \hat{\phi}(\theta^*) = \mathbb{E}[\psi(\theta^*)] \psi(\theta^*) \), and, by specializing this latter expression to the \( z \) component, we obtain Eq. (16) which implies Eq. (15) as we have shown before.

Turn now to consider the term \( \sqrt{N}(\hat{\epsilon}_N - z^0) \) and note that it is equal to \( \sqrt{N}(\hat{\epsilon}_N - z^0) \) (in fact \( z^0 = z^* \)).

We show that
\[ \sqrt{N}(\hat{\epsilon}_N - z^0) \sim \mathcal{N}(0, 2x^0 \hat{\phi}(x^*, z^*)^{-1}). \quad \quad \quad \quad (17) \]

As a matter of fact, consider the following Taylor expansion (which holds almost surely thanks to the differentiability properties of transfer function coefficients in Assumption 1):
\[ 0 = \sqrt{N} \frac{\partial}{\partial z} V_N (\hat{\epsilon}_N, \hat{\epsilon}_N) \]
\[ = \sqrt{N} \frac{\partial}{\partial z} V_N (\hat{\epsilon}_N, \hat{\epsilon}_N) + \frac{\partial^2}{\partial z^2} V_N (\hat{\epsilon}_N, \hat{\epsilon}_N) \sqrt{N}(\hat{\epsilon}_N - z^*), \quad \quad \quad \quad (18) \]

where \( \hat{z}^* \) is a point between \( \hat{z}_N \) and \( z^* \) and the first equality follows from the fact that \( \hat{\theta}_N = [(\hat{z}_N)^T (\hat{z}_N)^T] \) is a minimizer of \( V_N \).

Then, we can follow the same rationale as in Ljung (1999a, Chapter 9) to conclude that:
- \( \sqrt{N}(\partial^2 \hat{\phi}/\partial z^2) V_N (\hat{\epsilon}_N, z^*) \sim \mathcal{N}(0, 2x^0 \hat{\phi}(x^*, z^*)^{-1}) \) (this result follows along the same lines as Theorem 9.1 in Ljung (1999a))
- \( (\partial^2 \hat{\phi}/\partial z^2) V_N (\hat{\epsilon}_N, \hat{z}^*) \rightarrow \hat{V}(x^*, z^*) \) almost surely (again, this result follows from Theorem 2B.1 in Ljung (1999a)).
These two facts imply (17) (see [Ljung, 1999a, chapter 9, for details]).

Equation (14) now follows from (17) and (15). Indeed, the left-hand side of (14) can be rewritten as (note that $z^* = z^0$)

$$
\left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\hat{e}_i(x_i) - \hat{e}_i(x_i)^\top) \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\hat{e}_i(x_i) - \hat{e}_i(x_i)^\top) \right] \right\}^{1/2},
$$

(19)

where the first term goes to zero (recall that $x_N \to \infty$) and the second one converges to a $\chi^2$-distributed random variable.

We next prove Eq. (13).

Note that the proof of (13) is substantially different from the one of (14) since $x^0 \neq x^*$ and, in contrast to $\hat{z}_N - z^0$, $\hat{x}_N - x^0$ does not tend to zero.

We commence by observing that, since $\hat{P}(x^*)$, has a constant value in the $x$ direction—recall that $\{(x', z')^\top | x', z': \Theta\}$ is the set of minimizers of $\hat{P}(\tilde{y})$—it holds that $\hat{P}_{xx}(x^*, z^*) = 0$, $\forall x$: $(x^*, x^0) = \text{an} \text{interior} \text{point of } \Theta$ and, in particular, $\hat{P}_{xx}(x^*, z^*) = 0$. On the other hand, proceeding as for (15), it can be proved that $(1/N) \sum \hat{e}_i(\hat{x}_N, \hat{z}_N) e_i(\hat{x}_N, \hat{z}_N)^\top \to \frac{1}{2} \hat{P}_{xx}(x^*, z^*)$ almost surely, and, thus,

$$
\frac{1}{N} \sum \hat{e}_i(\hat{x}_N, \hat{z}_N) e_i(\hat{x}_N, \hat{z}_N)^\top \to 0 \text{ almost surely.}
$$

This last equation suggests that Eq. (13) can be proved by characterizing the rate of convergence to 0 of $(1/N) \sum \hat{e}_i(\hat{x}_N, \hat{z}_N) e_i(\hat{x}_N, \hat{z}_N)^\top$.

Consider the following Taylor expansion:

$$(\hat{x}_N - x^0)^\top e_i(\hat{x}_N, \hat{z}_N)^2 = (\hat{x}_N - x^0)^\top e_i(\hat{x}_N, \hat{z}_N)^2 + (\hat{x}_N - x^0)^\top \frac{\partial}{\partial z} e_i(\hat{x}_N, \hat{z}_N)^2$$

as required in Assumption 1.

The convergence to zero in probability of (24) now follows similarly to the convergence to zero in probability of (19). As a matter of fact, the only difference between (24) and (19) stays in their kernel, where the kernel of (19)

$$
(1/N) \sum \hat{e}_i(\hat{y}_N) e_i(\hat{y}_N)^\top / \sqrt{N}
$$

(24)

The convergence to zero in probability of (24) now follows similarly to the convergence to zero in probability of (19). As a matter of fact, the only difference between (24) and (19) stays in their kernel, where the kernel of (19)

$$
(1/N) \sum \hat{e}_i(\hat{y}_N) e_i(\hat{y}_N)^\top / \sqrt{N}
$$

almost surely, as it follows from Theorem 2B.1 in [Ljung (1999a)].

This concludes the proof. □
A.2. Proof of Theorem 4

The asymptotic cost criterion can be rewritten through Parseval identity as
\[
\hat{V}(\vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(e^{-j\omega}, \vartheta) - G(e^{-j\omega}, \vartheta^0)|^2}{|H(e^{-j\omega}, \vartheta)|^2} F_\nu(d\omega) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{-j\omega}, \vartheta)|^2}{|H(e^{-j\omega}, \vartheta^0)|^2} \lambda^2 d\omega,
\]
where \(F_\nu\) is the spectral measure of \(u(t)\).

Let \(\vartheta^*\) be a minimizer of \(\hat{V}(\vartheta)\). Since also \(\vartheta^0\) minimizes \(\hat{V}(\vartheta)\), we have that
\[
\hat{V}(\vartheta^*) = \hat{V}(\vartheta^0) = \lambda^2.
\]
Thus,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(e^{-j\omega}, \vartheta^*) - G(e^{-j\omega}, \vartheta^0)|^2}{|H(e^{-j\omega}, \vartheta^0)|^2} F_\nu(d\omega) = 0 \quad (25)
\]
and
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{-j\omega}, \vartheta^0)|^2}{|H(e^{-j\omega}, \vartheta^0)|^2} \lambda^2 d\omega = \lambda^2. \quad (26)
\]
Eq. (26) implies that
\[
H(e^{-j\omega}, \vartheta^*) = H(e^{-j\omega}, \vartheta^0), \quad \forall \omega \in [0, \pi]. \quad (27)
\]
On the other hand, from Eq. (25) it follows that \(G(e^{-j\omega}, \vartheta^*)\) must be equal to \(G(e^{-j\omega}, \vartheta^0)\) at every frequency where \(u(t)\) is exciting. That is
\[
G(e^{-j\omega}, \vartheta^*) = G(e^{-j\omega}, \vartheta^0), \quad \forall \omega; F_\nu(A) > 0, \text{ for any open } A \text{ containing } \omega. \quad (28)
\]
Now, letting \(H(e^{-j\omega}, \vartheta) = N_H(e^{-j\omega}, \vartheta)/D_H(e^{-j\omega}, \vartheta)\) and \(G(e^{-j\omega}, \vartheta) = N_G(e^{-j\omega}, \vartheta)/D_G(e^{-j\omega}, \vartheta)\), Eqs. (27) and (28) can be rewritten as
\[
N_H(e^{-j\omega}, \vartheta^*) D_H(e^{-j\omega}, \vartheta^0) = D_H(e^{-j\omega}, \vartheta^*) N_H(e^{-j\omega}, \vartheta^0), \quad \forall \omega \in [0, \pi] \quad (29)
\]
and
\[
N_G(e^{-j\omega}, \vartheta^*) D_G(e^{-j\omega}, \vartheta^0) = D_G(e^{-j\omega}, \vartheta^*) N_G(e^{-j\omega}, \vartheta^0), \quad \forall \omega; F_\nu(A) > 0, \text{ for any open } A \text{ containing } \omega. \quad (30)
\]
For any fixed \(\omega\), these equations are linear in \(\vartheta^*\), so defining an affine subspace. Since the intersection of affine subspaces is an affine subspace, the set of \(\vartheta^*\) satisfying Eqs. (29) and (30) is still an affine subspace. This concludes the proof. \(\square\)

A.3. Proof of Theorems 5

Let us consider the ARMAX case first.

Define: \(G^0(z^{-1}) = B^0(z^{-1})/A^0(z^{-1})\), \(H^0(z^{-1}) = C^0(z^{-1})/A^0(z^{-1})\), \(G(z^{-1}) = B(z^{-1})/A(z^{-1})\) and \(H(z^{-1}) = C(z^{-1})/A(z^{-1})\), where \(A^0(z^{-1})\), \(B^0(z^{-1})\) and \(C^0(z^{-1})\) stand for \(A(z^{-1}, \vartheta^0)\), \(B(z^{-1}, \vartheta^0)\) and \(C(z^{-1}, \vartheta^0)\), respectively.

Similarly to the proof of Theorem 4, the asymptotic cost criterion can be rewritten through Parseval identity as
\[
\hat{V}(\vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(e^{-j\omega}, \vartheta) - G^0(e^{-j\omega})|^2}{|1 + R(e^{-j\omega})G^0(e^{-j\omega})|^2} \lambda^2 d\omega
\]
\[
\times \frac{|R(e^{-j\omega})|^2}{|H(e^{-j\omega}, \vartheta)|^2} F_\nu(d\omega)
\]
\[
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|H(e^{-j\omega}, \vartheta)|^2}{|H(e^{-j\omega}, \vartheta^0)|^2} \lambda^2 d\omega
\]
\[
\times \frac{|1 + R(e^{-j\omega})G(e^{-j\omega}, \vartheta)|^2}{|1 + R(e^{-j\omega})G^0(e^{-j\omega})|^2} \lambda^2 d\omega,
\]
where \(F_\nu\) is the spectral measure of \(u(t)\).

Now, following the same rationale as in the proof of Theorem 4, we obtain that \(\vartheta^*\) is a minimizer of \(\hat{V}(\vartheta)\) if and only if
\[
G(e^{-j\omega}, \vartheta^*) - G^0(e^{-j\omega}) = 0,
\]
\[
\forall \omega; F_\nu(A) > 0, \text{ for any open } A \text{ containing } \omega. \quad (31)
\]
and
\[
\frac{H^0(e^{-j\omega})}{H(e^{-j\omega}, \vartheta^*)} \cdot \frac{1 + R(e^{-j\omega})G^0(e^{-j\omega})}{1 + R(e^{-j\omega})G(e^{-j\omega}, \vartheta^*)} = 1, \forall \omega \in [0, \pi]. \quad (32)
\]
Then, by the definition of \(G, G^0, H\) and \(H^0\) we have that (the dependencies on \(\vartheta\) and \(e^{-j\omega}\) have been omitted to ease the notation):
\[
G - G^0 = B^0 - \frac{B}{A^0},
\]
and
\[
\frac{H^0}{H} \cdot \frac{1 + R G}{1 + R G^0} = \frac{C^0 A}{A^0 C} \cdot \frac{D_{RA} + N_R B}{D_{RA}} \cdot \frac{D_{RA}^0}{D_{RA}^0 + N_R B^0} = \frac{C^0}{C} \cdot \frac{D_{RA} + N_R B}{D_{RA}^0 + N_R B^0},
\]
where \(N_R\) and \(D_{RA}\) are, respectively, the numerator and the denominator of \(R\).

As a consequence, Eqs. (31) and (32) can be rewritten as
\[
B(e^{-j\omega}, \vartheta^*) A^0(e^{-j\omega}) = B^0(e^{-j\omega}) A(e^{-j\omega}, \vartheta^*),
\]
\[
\forall \omega; F_\nu(A) > 0, \text{ for any open } A \text{ containing } \omega. \quad (33)
\]
and
\[
C^0(e^{-j\omega}) (D_R(e^{-j\omega}) A(e^{-j\omega}, \vartheta^*) + N_R(e^{-j\omega}) B(e^{-j\omega}, \vartheta^*))
\]
\[
= C(e^{-j\omega}, \vartheta^*) (D_R(e^{-j\omega}) A^0(e^{-j\omega})
\]
\[
+ N_R(e^{-j\omega}) B^0(e^{-j\omega}), \forall \omega \in [0, \pi]. \quad (34)
\]
As in Theorem 4, for any fixed $\omega$ these equations are linear in $\hat{\theta}$ and, therefore, the set of $\hat{\theta}$ satisfying Eqs. (33) and (34) is an affine subspace.

The same proof applies also for OE models considering $G(z^{-1},\hat{\theta}) = B(z^{-1},\hat{\theta})A(z^{-1},\hat{\theta})^{-1}$, $G^0(z^{-1}) = B^0(z^{-1})A^0(z^{-1})$ and $H^0(z^{-1}) = H(z^{-1},\hat{\theta}) = 1$ in this case.

References


Simone Garatti was born in Brescia, Italy, in 1976. He received the Laurea degree and the Ph.D. in Information Technology Engineering in 2000 and 2004, respectively, both from the Politecnico di Milano, Milano, Italy. Currently, he is a Research Assistant at the Dipartimento di Elettronica ed Informazione of the Politecnico di Milano. His research interests include system identification and model quality assessment, data-mining and iterative control.

Marco Claudio Campi is Professor of Automatic Control at the University of Brescia, Italy.

He was born in Tradate, Italy, on December 7, 1963. In 1988, he received the Doctor degree in Electronic Engineering from the Politecnico di Milano, Milano, Italy (his doctoral thesis was awarded the “Giorgio Quazza” prize as the best original thesis for year 1988). From 1988 to 1989, he was a Research Assistant at the Department of Electrical Engineering of the Politecnico di Milano. From 1989 to 1992, he worked as a Researcher at the Centro di Teoria dei Sistemi of the National Research Council (CNR) in Milano. Since 1992, he has been with the University of Brescia, Italy.

M.C. Campi is an Associate Editor of Automatica and Systems and Control Letters, and a past Associate Editor of the European Journal of Control. Serves as Chair of the Technical Committee IFAC on Stochastic Systems (SS) and is a member of the Technical Committee IFAC on Modeling, Identification and Signal Processing (MISP). Moreover, he is a Distinguished Lecturer under the IEEE Control System Society (CSS) Program.

He has held visiting and teaching positions at many universities and institutions including the Australian National University, Canberra, Australia, the University of Illinois at Urbana-Champaign, USA, the Centre for Artificial Intelligence and Robotics, Bangalore, India, and the University of Melbourne, Australia.

The research interests of M.C. Campi include: adaptive and data-based control, system identification, robust convex optimization, robust control and estimation, and learning theory. His research activity has been conducted through years under many Italian and European projects. Currently, he is leader of the Brescia unit of the European IST project “Distributed control and stochastic analysis of hybrid systems supporting safety critical real-time systems design”. The research activity of M.C. Campi is witnessed by more than 40 papers published in archival journals.
Sergio Bittanti (b. 1947–EE 1970; MATH 1978) is Professor in the Milan Institute of Technology (Politecnico di Milano—Italy). He has been working in the Department of Electronics and Information Sciences of that University since 1970. In his early academic days, he had the privilege of serving as Assistant of Giorgio Quazza for his exams of Process Control.

Besides being active on theoretical topics of system identification and control, he has developed a number of advanced application studies with industries and various Institutions. The width of his interests in science and engineering is witnessed by the outcomes of this intense research activity as summarized in more than 120 refereed papers, published not only in the best journals of control and automation, but also in top journals of aeronautics, nuclear science, oceanic engineering, semiconductor manufacturing, laser and particle beams, to quote only a few. He also contributed more than 150 papers to International Conferences.

Sergio Bittanti is the author of a number of textbooks in Italian and the Editor or Co-editor of four volumes in English (all with Springer-Verlag). In particular, together with Giorgio Picci, he is the Editor of Identification, Adaptation, Learning—The Science of Learning Models from Data (1996), which contains a collection of selected lectures given at the NATO ASI From Identification to Learning (Como, Italy, 1994). Moreover, together with A.J. Laub and J.C. Willems, he is Editor of The Riccati Equation (1991), a book published after the IFAC/IEEE/SIAM workshop on The Riccati Equation in Control, Systems, and Signals (Como, Italy, 1989).

Besides the previously mentioned events, he also served as Chairman for the following conferences: IFAC Symposium on Robust Control Design (Milan, 2003, NOC Chair), IFAC Workshop on Adaptation and Learning in Control and Signal Processing (Cernobbio-Como, 2001, IPC Chair), IFAC Workshop on Periodic Control Systems (Cernobbio, Como, 2001, IPC Chair) and Workshop on New Horizons in System Theory (Como, Italy, 1986, Co-chair R.E. Kalman). He also substantially contributed to the Third European Control Conference (Rome, Summer 1995).

Sergio Bittanti is the Editor in Chief of the European Journal of Control; moreover, he acted or acts as Associate Editor for: Annual Reviews in Control; Journal of Mathematical Systems, Estimation, and Control; Statistics and Computing; European Journal of Control.

He is also a Fellow of IEEE. Within IFAC, he covered various positions; in particular he was the Chairman of the Technical Committee Adaptation and Learning in Control and Signal Processing; presently, he is a member of the IFAC Council.

He has been involved in various European Projects, among which the “Non-linear and Adaptive Control” (NACO) TMR network. For several years he acted as project manager of the Italian research network Model Identification, System Control and Signal Processing, which connected about 70 Italian professor and researchers distributed in different Universities. Also, for long time he served as supervisor of the Ph.D. Program in Informatica & Automatica at the Politecnico di Milano. Finally, he is a long-term Associate Fellow to the Italian Research Council (Consiglio Nazionale delle Ricerche).