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Virtual Reference Feedback Tuning for industrial PID controllers ⋆

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Abstract: In this paper, we propose a data-based auto-tuning method for industrial PID controllers, which does not rely on a model of the plant. The method is inspired by the Virtual Reference Feedback Tuning approach for data-based controller tuning, but it is taylored to the framework of PID controller design. The method is entirely developed in a deterministic, continuous time setting, where the assumption of stationarity is not needed. The effectiveness of the proposed approach is tested on a benchmark example that has been recently proposed for the evaluation of PID controllers.

Keywords: VRFT, PID, data-based control.

1. INTRODUCTION

Proportional-Integral-Derivative (PID) control technology is the most widely used approach for feedback regulation of automatic systems. It is estimated that, nowadays, more than 90% of practical control systems employs such a technology (see Li et al. [2006a]). The reasons are many, but certainly one of the most significant motivations is the fact that, with only three parameters, a PID controller can effectively fulfill the most common requirements of typical industrial control problems, ranging, *e.g.*, from zero steady-state error in tracking a constant setpoint to disturbance rejection (see Li et al. [2006b]).

Over the years, the everlasting popularity of PID controllers has led to the development of several tuning methods, which try to offer effective strategies that are also fast and simple. More specifically, due to the need of fast recalibration of existing systems, both academic and industrial people have dedicated a significant amount of time and effort in the development of PID autotuning methods, the first of them being the Ziegler and Nichols approach in Ziegler and Nichols [1942].

Since 1942, several approaches to PID auto-tuning have been proposed and applied to a huge number of different applications. For an overview of the main scientific approaches, see Vilanova and Visioli [2012], whereas Kocijan [2008] presents an accurate survey of the methods for PID auto-tuning proposed as patents. From such readings, it becomes evident how large the world of available techniques is, and that the assessment of one method with respect to the others is deeply related to the specific application, as also observed and discussed in Leva and Donida [2009].

A complete taxonomy of PID auto-tuning methods is difficult to draw up here. However, it can be noted that the majority of the existing approaches is either characterized by the use of semi-empirical rules or derived from model-based methods employing low-order data-based models (see again Li et al. [2006b]). This observation is very important, because it means that strong guarantees on the real system cannot be provided: on the one hand, semi-empirical rules are not based on optimization theories and one can only hope to obtain a suitable tuning for the application at hand; on the other hand, low-order models are always approximate, and modeling errors might strongly jeopardize the performance of the closed-loop system (see Hjalmarsson [2005]).

To overcome these problems, in the last decades people in systems and control have investigated data-based controller tuning techniques aimed to design suitable feedback controllers directly from data without the need of identifying a model of the system. The advantages of such a change of perspective are evident, and the last ten years gave birth to a large variety of methods, among which Iterative Feedback Tuning (IFT, see Hjalmarsson et al. [1998]), data-driven loop-shaping (see Formentin and Karimi [2013]), Virtual Reference Feedback Tuning (VRFT, see Campi et al. [2002], Lecchini et al. [2002], Campi et al. [2003]) and Correlation-based Tuning (CbT, see van Heusden et al. [2011]). In particular, the last two methods, which are based on non-iterative optimization, showed to be comparable - in terms of statistical performance - to standard model-based design approaches from data (see Formentin et al. [2013b]), since the control problem can be completely recast into a system identification problem. This is a great advantage in that a lot of established results for improving estimation performance can be employed also to improve the performance of controller design, see, *e.g.*, optimal input design in Formentin et al. [2013a].

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VRFT and CbT have been developed in the stochastic set-up described in Ljung [1999], where the involved processes are stationary and evolve in discrete-time. In contrast to the previous studies, the aim of this work is to reformulate the VRFT method introduced in Campi et al. [2002], so as to better fit into the framework of PID control design for industrial use. Specifically, in this novel form, Input/Ouput (I/O) signals are not treated as stochastic processes and the theory can be fully interpreted in continuous time. A first attempt to adapt the VRFT method to PID controller design has been proposed, within an internal model control (IMC) framework, in Rojas and Vilanova [2012].

The outline of the paper is as follows. In Section 2, we will first present the algorithm with the tuning rules for the PID gains. The theoretical details behind the approach will be given in Section 3, but the reading of this section is not necessary for the understanding of the rest of the paper. We then provide an illustrative example in Section 4 and some concluding remarks in Section 5. The proofs of the theorems will be omitted for space limitations; however, they are available from the authors upon request.

2. ALGORITHM FOR THE USER

Let $\mathcal{G}$ be the plant to be controlled. Suppose that $\mathcal{G}$ is open-loop stable or that it is operated in closed-loop with a (possibly poorly performing) stabilizing PID controller.

The desired frequency behaviour for the closed-loop system is described as $M(j\omega)$, where $M(j\omega)$ is given, or, alternatively, it is derived from user requirements. For example, if a desired settling time of $1s$ is required, the frequency response $M(j\omega) = 1/(j0.2\omega + 1)$ can be used. In any case, let $M(j0) = 1$. Assume also that the closed-loop model matching has to be weighted according to the frequency weight $W(j\omega)$. If no weighting is provided, take $W(j\omega) = 1$, $\forall\omega$.

Finally, consider the frequency response of the PID controller $C(j\omega, \rho) = \rho^T \beta(j\omega)$, where $\rho = [K_p \ K_i \ K_d]^T$ are the parameters to tune and

$$\beta(j\omega) = \left[1, \ \frac{1}{j\omega} \ \frac{j\omega}{1 + j\omega T_d}\right]^T,$$

and select the time constant of the derivative part T_d as twice the sampling time.

The Virtual Reference Feedback Tuning algorithm to design the PID gains is then as follows.

VRFT algorithm (given M, W, T_d, stability of $\mathcal{G}$)

(1) **[Input selection]** If $\mathcal{G}$ is open-loop stable (CASE A), set an input sequence $u(t)$. Alternatively (CASE B), set a reference sequence $r(t)$.
(2) **[Experiment]** Run the experiment on the real system, using $u(t)$ in open-loop (CASE A) or $r(t)$ in closed-loop (CASE B). Record the input signal $u(t)$ (or $r(t)$) and the corresponding output signal $y(t)$ from the starting time t_0 to the final instant t_f.
(3) **[Virtual signals for controller identification]** Compute the *virtual reference* signal $r_v(t)$ as the output of $M^{-1}(j\omega)$ [1] when it is fed by $y(t)$ and the *virtual error* signal $e_v(t)$ as $e_v(t) = r_v(t) - y(t)$.
(4) **[Filter selection]** Select a filter $\mathcal{F}$ such that
$$F(j\omega)U(\omega) = M(j\omega)\left(1 - M(j\omega)\right)W(j\omega), \ \forall\omega,$$
holds, where $U(\omega)$ is the Fourier transform of $u(t)$ (see Gröchenig [2001]). Then, filter the controller I/O signals as $U_F(\omega) = F(j\omega)U(\omega)$ and $E_F(\omega) = F(j\omega)E_v(\omega)$ to get $u_F(t)$ and $e_F(t)$.
(5) **[Pre-processing]** Build the regressors:
$$\varphi_F(t) = [\varphi_1(t), \ \varphi_2(t), \ \varphi_3(t)]^T, \tag{1}$$
where
$$\varphi_1(t) = e_F(t), \ \varphi_2(t) = \int_{t_0}^{t_f} e_F(t) \ dt,$$
and $\varphi_3(t)$ is the output of $\Phi(j\omega) = j\omega/(j\omega T_d + 1)$ when it is fed by $e_F(t)$.
(6) **[Instrumental variable]** Identify a (possibly inaccurate) low-order model $\hat{\mathcal{G}}$ between $u(t)$ and $y(t)$. Compute $\hat{y}(t)$ as the simulated output of $\hat{\mathcal{G}}$ when the input is $u(t)$, and the corresponding $\hat{e}_F(t)$. Then, define the instrumental variable $\xi(t) = \hat{\varphi}_F(t)$, where $\hat{\varphi}_F(t)$ is computed according to (1), but using $\hat{e}_F(t)$ instead of $e_F(t)$.
(7) **[PID tuning]** Compute the PID gains as in (2), where
$$x_{ij} = \int_{-\infty}^{\infty} \xi_i(t)\varphi_j(t)dt, \ i, j = 1, 2, 3,$$
$$x_{iu} = \int_{-\infty}^{\infty} \xi_i(t)u_F(t)dt, \ i = 1, 2, 3$$
and $\xi_i(t)$ and $\varphi_j(t)$ denote, respectively, the i^{th} element of $\xi(t)$ and the j^{th} element of $\varphi_F(t)$.

In the following sections, we will show that the above algorithm allows the resulting PID controller achieves the desired closed-loop behavior when it is possible and, in any case, minimizes a suitable frequency-wise model matching error (also when the output signal is corrupted by some measurement noise).

3. THE THEORY BEHIND

In this section, we provide the assumptions and the technical details behind the algorithm presented in Section 2. This section is intended for the interested reader and its reading is not necessary for the understanding of the rest of the paper.

3.1 Preliminaries on Signals and Systems

A complex-valued function $x(t)$ is said to be in $\mathcal{L}^2$ if it is measurable and square-integrable, *i.e.*,
$$\int_{-\infty}^{\infty} |x(t)|^2 \ dt < \infty.$$
In this paper, a *signal* is always a real-valued function that belongs to $\mathcal{L}^2$. According to Plancherel theorem, see Gröchenig [2001], each $x(t) \in \mathcal{L}^2$ has a Fourier

[1] As it is reasonable for a tracking control problem, we assume $M(j\omega) \neq 0$ on the imaginary axis. If the roots of the numerator of $M(j\omega)$ have positive real parts, we filter $y(t)$ backward in time. This is possible due to that $y(t)$ is processed off-line.

TUNING RULES

$$K_p^{vr} = \frac{x_{23}^2 x_{1u} - x_{12}x_{23}x_{3u} + x_{13}x_{22}x_{3u} - x_{13}x_{23}x_{2u} + x_{12}x_{33}x_{2u} - x_{22}x_{33}x_{1u}}{x_{33}x_{12}^2 - 2x_{12}x_{13}x_{23} + x_{22}x_{13}^2 + x_{11}x_{23}^2 - x_{11}x_{22}x_{33}} \quad (2a)$$

$$K_i^{vr} = \frac{x_{13}^2 x_{2u} - x_{12}x_{13}x_{3u} + x_{11}x_{22}x_{3u} - x_{13}x_{23}x_{1u} - x_{11}x_{33}x_{2u} + x_{12}x_{33}x_{1u}}{x_{33}x_{12}^2 - 2x_{12}x_{13}x_{23} + x_{22}x_{13}^2 + x_{11}x_{23}^2 - x_{11}x_{22}x_{33}} \quad (2b)$$

$$K_d^{vr} = \frac{x_{12}^2 x_{3u} - x_{12}x_{13}x_{2u} - x_{11}x_{22}x_{3u} + x_{11}x_{23}x_{2u} - x_{12}x_{23}x_{1u} - x_{13}x_{22}x_{1u}}{x_{33}x_{12}^2 - 2x_{12}x_{13}x_{23} + x_{22}x_{13}^2 + x_{11}x_{23}^2 - x_{11}x_{22}x_{33}} \quad (2c)$$

transform $X(\omega) \in \mathcal{L}^2$, indicated with the capital letter, and, viceversa, each $X(\omega) \in \mathcal{L}^2$ is the Fourier transform of a function $x(t) \in \mathcal{L}^2$.

Given $x(t) \in \mathcal{L}^2$, if the distributional derivative of $x(t)$ of order i, written $x^{(i)}(t)$, is in $\mathcal{L}^2$ for all $i \leq p$, then $x(t)$ is said to belong to $\mathcal{H}^p$, the Sobolev space of order p (see Adams and Fournier [2003]).

Consider now the linear Ordinary Differential Equation (ODE)

$$\mathcal{Z}: \sum_{i=0}^{n} \alpha_i v^{(i)}(t) - \sum_{i=0}^{m} \beta_i x^{(i)}(t) = 0.$$

An equation like $\mathcal{Z}$ represents the dynamics of a *system*.

For notational convenience, introduce the polynomials

$$A(j\omega) = \sum_{i=0}^{n} \alpha_i (j\omega)^i, \quad B(j\omega) = \sum_{i=0}^{m} \beta_i (j\omega)^i.$$

The ratio $Z(j\omega) = B(j\omega)/A(j\omega)$ is called the *frequency response* of $\mathcal{Z}$.

We assume that $\alpha_n \neq 0$ (this is not a real condition, it simply means that the largest derivative of $v(t)$ in the ODE has order n), and, for the time being, also assume that $A(j\omega) \neq 0$ on the imaginary axis (this condition is removed later). Note that this condition does not prevents the ODE to be unstable, that is, the roots of the numerator of $A(j\omega)$ can have positive real parts. If $x(t) \in \mathcal{H}^m$, then $\sum_{i=0}^{m} \beta_i x^{(i)}(t)$ is in $\mathcal{L}^2$, and, by Plancherel theorem, it has Fourier transform $B(j\omega)X(\omega) \in \mathcal{L}^2$. Consider $V(\omega) := [B(j\omega)/A(j\omega)]X(\omega)$. Clearly, $V(\omega)$ is in $\mathcal{L}^2$, so that, based again on Plancherel theorem, it is the Fourier transform of a function $v(t) \in \mathcal{L}^2$. In fact, we know more, that is, $v(t) \in \mathcal{H}^n$. This follows by applying Plancherel theorem to $(j\omega)^i V(\omega)$, $i \leq n$, the Fourier transform of the ith distributional derivative of $v(t)$, which is in $\mathcal{L}^2$.

We claim that this $v(t)$ is *the only solution in* $\mathcal{L}^2$ *of the ODE* $\mathcal{Z}$. To show this, note that $A(j\omega)V(\omega)$ is the Fourier transform of $\sum_{i=0}^{n} \alpha_i v^{(i)}(t)$. On the other hand, $A(j\omega)V(\omega) = B(j\omega)X(\omega)$, which is the Fourier transform of $\sum_{i=0}^{m} \beta_i x^{(i)}(t)$. Thus, $\sum_{i=0}^{n} \alpha_i v^{(i)}(t) = \sum_{i=0}^{m} \beta_i x^{(i)}(t)$ and $v(t)$ satisfies the ODE $\mathcal{Z}$. It is in fact the only solution in $\mathcal{L}^2$ of $\mathcal{Z}$ since any other solution is obtained by adding to this $v(t)$ a linear combination of the modes of $\mathcal{Z}$, which is not a function of $\mathcal{L}^2$.

As a short-hand, throughout the paper given $x(t) \in \mathcal{H}^m$ the only solution of $\mathcal{Z}$ in $\mathcal{L}^2$ is written as $\mathcal{Z}[x(t)]$.

The above reasoning extends to when $A(j\omega)$ annihilates on the imaginary axis, provided that these zeros are canceled by $X(\omega)$ so that $[B(j\omega)/A(j\omega)]X(\omega)$ is in $\mathcal{L}^2$. As an example, suppose that the ODE is the integrator

$$v^{(1)}(t) - x(t) = 0,$$

and $x(t) = sgn(t)e^{-|t|}$ (which has 0 dc-component), then $X(\omega) = -2j\omega/(1+\omega^2)$ and $[B(j\omega)/A(j\omega)]X(\omega) = -2/(1+\omega^2) \in \mathcal{L}^2$, which corresponds to $v(t) = -e^{-|t|}$.

3.2 Problem formulation

Let the *plant dynamics* be described by the linear ODE

$$\mathcal{G}: \sum_{i=0}^{n_a} a_i y^{(i)}(t) - \sum_{i=0}^{n_b} b_i u^{(i)}(t) = 0. \quad (3)$$

We assume that $u(t)$ belongs to suitable Sobolev spaces, such that $y(t)$, and all the other signals derived from it, are in $\mathcal{L}^2$.

The *PID controller* is instead described by

$$\begin{aligned}\mathcal{C}_\rho: \ & T_d u^{(2)}(t) + u^{(1)}(t) - (K_d + K_p T_d) e^{(2)}(t) + \\ & - (K_p + K_i T_d) e^{(1)}(t) - K_i e^{(0)}(t) = 0, \quad (4)\end{aligned}$$

where $e(t) = r(t) - y(t)$, $r(t)$ is the reference signal, $\rho = [K_p,\ K_i,\ K_d]^T$ is the vector of tuning parameters and T_d is fixed. The frequency response of $\mathcal{C}_\rho$ is

$$C(j\omega, \rho) = \rho^T \beta(j\omega), \quad (5)$$

where

$$\beta(j\omega) = \left[1,\ \frac{1}{j\omega},\ \frac{j\omega}{1 + j\omega T_d}\right]^T.$$

In (5), the classical PID controller structure can be clearly distinguished.

Finally, let $\mathcal{M}$ be an ODE representing the *desired closed-loop system* and $M(j\omega)$ its frequency response such that $M(j0) = 1$. Moreover, let $W(j\omega)$ be a user-defined frequency weight.

We can now formally define the model-reference control problem addressed in this work.

Problem 1. (Model-reference design). Find a controller of the form in (4) minimizing the *model-reference cost function*

$$J_{mr}(\rho) = \int_{-\infty}^{+\infty} \left| \left(\frac{G(j\omega)C(j\omega,\rho)}{1 + G(j\omega)C(j\omega,\rho)} - M(j\omega) \right) W(j\omega) \right|^2 d\omega. \quad (6)$$

Definition 1. (Optimal controller). The optimal solution to Problem 1, assuming that it exists and is unique, is the PID controller $\mathcal{C}^o$ with frequency response

$$C(j\omega, \rho_o) = \rho_o^T \beta(j\omega), \quad (7)$$

where

$$\rho_o = \arg\min_{\rho} J_{mr}(\rho).$$

In most of the cases, $\mathcal{C}^o$ does not yield $J_{mr}(\rho) = 0$, due to that the controller is constrained to belong to the PID class.

In many real-world applications, the frequency response $G(j\omega)$ of the plant is unavailable. To compensate for such a lack of knowledge, in this paper, it is assumed that we are allowed to run experiments on the plant $\mathcal{G}$ and we will use the collected I/O trajectories $u(t)$ and $y(t)$ to solve Problem 1, without a need of deriving a model of the plant.

3.3 The VRFT approach

In this subsection, we show how the controller design issue in Problem 1 can be recast into a data-based controller identification problem, which does not require a direct knowledge of $\mathcal{G}$. We will first deal with noise-free data; noisy data will be considered in the second part of the section. From now on, the argument $j\omega$ will be often omitted.

The main idea

Consider the "virtual" loop depicted in Figure 1, where $u(t)$ and $y(t)$ are the actual signals that are recorded during an experiment on the plant. The *virtual reference*

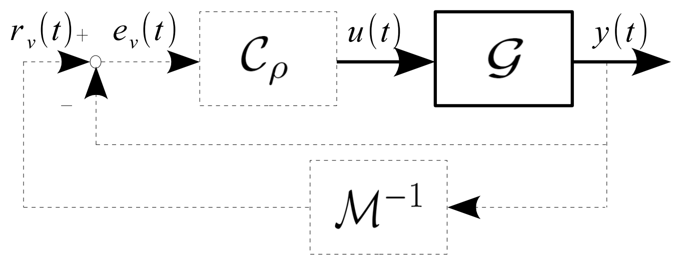

Fig. 1. The "virtual loop" and the real plant.

signal $r_v(t)$ and the *virtual error* signal $e_v(t)$ are computed respectively as

$$r_v(t) = \mathcal{M}^{-1}[y(t)],$$
$$e_v(t) = r_v(t) - y(t).$$

The relation $r_v(t) = \mathcal{M}^{-1}[y(t)]$ denotes the $\mathcal{L}^2$ signal generated by $\mathcal{M}$ when $y(t)$ is seen as the input signal.

The control design problem can then be reduced to an identification problem, where the system to be identified is the controller that generates $u(t)$ when it is fed by $e_v(t)$. Such a controller can be defined as follows.

Definition 2. (VR controller). The *VR controller* $\mathcal{C}_{vr}$ is the PID controller (4) with frequency response

$$C(j\omega, \rho_{vr}) = \rho_{vr}^T \beta(j\omega), \tag{8}$$

where

$$\rho_{vr} = \arg\min_{\rho} J_{vr}(\rho)$$

and $J_{vr}(\rho)$ is the *VR cost function.* Specifically,

$$J_{vr}(\rho) = \int_{-\infty}^{+\infty} \left(u_F(t) - C_\rho[e_F(t)]\right)^2 dt, \tag{9}$$

where

$$u_F(t) = \mathcal{F}[u(t)], \; e_F(t) = \mathcal{F}[e_v(t)]$$

and $\mathcal{F}$ is a data-prefilter with frequency response $F(j\omega)$.

From now on, $F(j\omega)$ is assumed to be a fractional rational function of its argument with maximum power of the numerator $n_f \leq n_u$. Moreover, being J_{vr} quadratic in ρ, by assuming that ρ_{vr} exists, the minimum point is unique.

Remark 1. Notice that the integral term in (9) requires infinite trajectories, whereas in reality $u(t)$ and $y(t)$ are known only over a finite observation time $[t_0, \; t_f]$. However, notice that the trajectory of any signal $x(t) \in \mathcal{L}^2$ can always be extended to $(-\infty, \; \infty)$, by assigning $x(t) = 0$, for all $t \notin [t_0, \; t_f]$.

□

Remark 2. Notice that, if $\mathcal{M}$ is characterized by non-minimum phase dynamics, $\mathcal{M}^{-1}$ is an unstable system and r_v might not be in $\mathcal{L}^2$. However, since r_v is computed off-line, the solution of $\mathcal{M}^{-1}[y(t)]$ can be integrated backward in time, thus leading to a stable filtering operation and $r_v(t) \in \mathcal{L}^2$.

□

For analysis purposes, let us now introduce the *extended controller* $\mathcal{C}_\rho^+$ as a controller, parameterized with $\rho^+ = [\rho^T, \; K_\delta]^T$, with frequency response

$$C^+(j\omega, \rho^+) = C(j\omega, \rho) + K_\delta \delta(j\omega), \tag{10}$$

where $\delta(j\omega)$ is the frequency response of a system δ such that

$$J_{mr}^+(\rho_o^+) = 0 \tag{11}$$

holds, being $\rho_o^+ = [\rho_o^T, \; 1]^T$ and

$$J_{mr}^+(\rho^+) = \int_{-\infty}^{+\infty} \left| \left(\frac{G(j\omega)C^+(j\omega,\rho^+)}{1 + G(j\omega)C^+(j\omega,\rho^+)} - M(j\omega) \right) W(j\omega) \right|^2 d\omega. \tag{12}$$

Under the assumption that ρ_o^+ exists and is unique, the ideal controller achieving $\mathcal{M}$ in closed-loop operation can be defined as follows.

Definition 3. (Ideal controller). The *ideal controller* $\mathcal{C}_o^+$ is an extended controller with frequency response

$$C^+(j\omega, \rho_o^+) = C(j\omega, \rho_o) + \delta(j\omega). \tag{13}$$

Under very mild assumptions on $\mathcal{F}$ - which come out in the proof of the theorem herein omitted - the following relationship between $\mathcal{C}_o$ and $\mathcal{C}_{vr}$ holds.

Theorem 1. (Model-Data equivalence). Assume that the ideal controller $\mathcal{C}_o^+$ is a PID, *i.e.* $\delta(j\omega) = 0, \; \forall \omega$ in (10). Then, for any selection of $\mathcal{F}$, it holds that

$$\rho_{vr} = \rho_o.$$

□

The result in Theorem 1 is fundamental in motivating the VRFT approach since, in the case where a PID controller is sufficient to achieve the desired closed-loop behavior, such a statement establishes the *theoretical equivalence* between model-based design (minimizing J_{mr}) and data-based controller identification (using the method introduced here).

However, since $\mathcal{G}$ is supposed to be unknown, the assumption on $\mathcal{C}_o^+$ in Theorem 1 cannot be verified in practice, nor can $\mathcal{M}$ be modified accordingly. Nevertheless, in the more realistic case where $\mathcal{C}_o^+$ is not a PID, we can still use the data prefilter $\mathcal{F}$ for bias shaping, in order to obtain a controller that closely resembles C_o. The following statement is helpful in deriving a relationship between $\mathcal{C}_o$ and $\mathcal{C}_{vr}$ and provides a suitable selection of $\mathcal{F}$.

Theorem 2. (Filter selection). Consider the general case where $\mathcal{C}_o^+$ is any controller. If the frequency response of the data prefilter $\mathcal{F}$ satisfies

$$F(j\omega)U(\omega) = M(j\omega)\left(1 - M(j\omega)\right)W(j\omega),\ \forall\omega, \tag{14}$$

it holds that

$$\rho_{vr} = \arg\min_{\rho} \bar{J}_{mr}^{+}([\rho,\ 0]^T),$$

where

$$\bar{J}_{mr}^{+}(\rho^+) = (\rho^+ - \rho_o^+)^T \left.\frac{\partial^2 J_{mr}^+}{\partial \rho^{+2}}\right|_{\rho_o^+} (\rho^+ - \rho_o^+). \tag{15}$$

□

From a practical perspective, Theorem 2 means that, although generally $\rho_{vr} \neq \rho_o$, when $\delta(j\omega)$ is small $\forall\omega$ and $\mathcal{F}$ is suitably selected, the minimizer of J_{vr} can approximately be rewritten as

$$\rho_{vr} = \arg\min_{\rho} \bar{J}_{mr}^{+}([\rho,\ 0]^T) \approx \arg\min_{\rho} \bar{J}_{mr}(\rho), \tag{16}$$

where

$$\bar{J}_{mr}(\rho) = (\rho - \rho_o)^T \frac{\partial^2 J_{mr}}{\partial \rho^2}(\rho - \rho_o),$$

that is,

$$\rho_{vr} \approx \rho_o.$$

Dealing with output noise

Consider now the case where the output data is corrupted by an additive noise $v(t) \in \mathcal{L}^2$, namely

$$y^v(t) = y(t) + v(t).$$

It can be shown that minimization of (9) using the noisy dataset does not lead to the previous minimizer $\rho_{vr} = \rho_o$. As a matter of fact, the virtual error signal corresponding to the new output is

$$e_F^v(t) = e_F(t) + e^v(t) = e_F(t) + \mathcal{M}^{-1}[v(t)] - v(t)$$

and the VR cost function becomes

$$\begin{aligned} J_{vr}^v(\rho) &= \int_{-\infty}^{+\infty} (u_F(t) - C_\rho[e_F^v(t)])^2\, dt \\ &= J_{vr}(\rho) + \int_{-\infty}^{+\infty} \left(C_\rho\left[\mathcal{M}^{-1}[v(t)] - v(t)\right]\right)^2 dt + \\ &- 2\int_{-\infty}^{+\infty} (u_F(t) - C_\rho[e_F(t)])\, C_\rho\left[\mathcal{M}^{-1}[v(t)] - v(t)\right] dt, \end{aligned}$$

which is generally not minimized by ρ_o, as the two additional terms are zero at $\rho = 0$ (*i.e.*, corresponding to the open loop configuration).

However, the tuning procedure can be suitably modified in order to obtain the noiseless estimate even in presence of noisy data.

To start with, build now the following regressors:

$$\begin{gathered} \varphi_F(t) = [\varphi_1(t),\ \varphi_2(t),\ \varphi_3(t)]^T, \\ \varphi_1(t) = e_F(t),\ \varphi_2(t) = \int_{-\infty}^{\infty} e_F(t)\ dt \\ \varphi_F^v(t) = \varphi_F(t) + \varphi^v(t), \\ \varphi^v(t) = [\varphi_1^v(t),\ \varphi_2^v(t),\ \varphi_3^v(t)]^T, \\ \varphi_1^v(t) = e^v(t),\ \varphi_2^v(t) = \int_{-\infty}^{\infty} e^v(t)\ dt \end{gathered} \tag{17}$$

where $\varphi_3(t)$ is the solution of

$$\varphi_3^{(1)}(t) = \frac{1}{T_d}\left(-\varphi_3(t) + e_F^{(1)}(t)\right)$$

and $\varphi_3^v(t)$ is the solution of

$$\varphi_3^{v(1)}(t) = \frac{1}{T_d}\left(-\varphi_3^v(t) + e^{v(1)}(t)\right).$$

Then, define the instrumental variable $\xi(t)$ as $\xi(t) = \varphi_F(t) + \varphi^\xi(t)$, where $\varphi^\xi(t)$ is a user-defined 3×3 matrix, and the estimate using noisy data as

$$\rho_{vr}^v = \left(\int_{-\infty}^{\infty} \xi(t)\varphi_F^{vT}(t)\ dt\right)^{-1} \int_{-\infty}^{\infty} \xi(t)u_F(t)\ dt. \tag{18}$$

We have the following result.

Theorem 3. (Noisy data). Assume that $u(t)$ and $v(t)$ satisfy

$$\int_{-\infty}^{+\infty} \varphi_F(t)\varphi^{vT}(t)\ dt = 0, \tag{19a}$$

$$\int_{-\infty}^{+\infty} \varphi_F(t)\varphi_F^T(t)\ dt > 0. \tag{19b}$$

If $\varphi^\xi(t)$ is selected such that

$$\int_{-\infty}^{+\infty} \varphi^\xi(t)\delta[e_F(t)]\ dt = 0, \tag{19c}$$

$$\int_{-\infty}^{+\infty} \varphi^\xi(t)\varphi_F^T(t)\ dt = 0, \tag{19d}$$

$$\int_{-\infty}^{+\infty} \varphi^\xi(t)\varphi^{vT}(t)\ dt = 0, \tag{19e}$$

then (18) yields the minimizer of the cost function $J_{vr}(\rho)$ with noiseless data, *i.e.*,

$$\rho_{vr}^v = \rho_{vr}.$$

□

Theorem 3 provides the most general tuning rule, according to which the optimal controller for the second order approximation of the extended cost function can be found for any selection of $\mathcal{M}$ and any value of the Signal-to-Noise ratio (SNR).

Recall that, when $\xi(t)$ is selected as suggested in Section 2, $\hat{\mathcal{G}}$ has to be computed. However, we could still say that the method is "model-free", since, as already mentioned, $\hat{\mathcal{G}}$ does not need to be an accurate model of the plant, but only to satisfy the assumptions of Theorem 3.

The PID gains given by the VRFT method can be *explicitly* computed, and Equations (2) (where $\varphi_F(t)$ is replaced by $\varphi_F^v(t)$ in case of noisy data) give the exact expressions.

4. SIMULATION EXAMPLE

The performance of the proposed design approach is evaluated on a benchmark system proposed in Morilla [2012] for PID tuning. The system describes a typical drum boiler as given in Åström and Bell [2000].

The function of a drum boiler is to deliver steam of a given quality (temperature and pressure) either to a single user, such as a steam turbine, or to a network of many users. For a proper functioning of the boiler, several requirements need to be satisfied, *e.g.* the ratio of air to fuel must be carefully controlled in order to obtain efficient combustion and the level of water in the drum must be controlled at the desired level to prevent overheating or flooding of steam lines. Finally, a desired steam pressure must be maintained at the outlet of the drum despite variations in the quantity of user steam demand. In this work, we will concentrate on the last objective, *i.e.* steam pressure control via fuel flow, and we will assume that the drum water level and the oxygen concentration are controlled independently by means of two additional loops.

To design the PID controller via the proposed method, the simple reference model

$$\mathcal{M}: \; y^{(1)}(t) = -0.033y(t) + 0.033r(t-17). \tag{20}$$

is considered, where the pure delay of 17 s can be easily found by means of a step excitation test. To obtain the data needed for tuning the controller, we inject in the simulator, as a fuel flow trajectory, a PRBS signal with mean value of 60% and amplitude of ±20%. Then, we collect the corresponding (noisy) steam pressure sequence $y(t)$. For the computation of the instrumental variable, an ARX model of order 2 is used.

The results of a closed-loop step test using the achieved PID controller are illustrated in Figure 2, where the VRFT controller is also compared to the desired response, *i.e.*, the step response of $\mathcal{M}$ in (20). In the same figure, the response with the benchmark controller embedded in the toolbox provided in Morilla [2012] is also shown.

It can be noticed that the VRFT controller shows sightly better performance than the benchmark PID with the advantage of being tunable using only one set of I/O data and without deriving a model of the drum boiler.

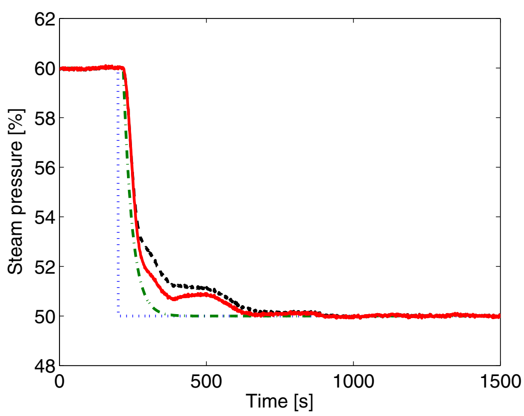

Fig. 2. Comparative test: step excitation (dotted), response of $\mathcal{M}$ (dash-dotted), PID of Morilla [2012] (dashed) and PID tuned via VRFT (solid).

5. CONCLUDING REMARKS

In this paper, the Virtual Reference Feedback Tuning method has been revisited to fit the framework of PID controller tuning. The proposed reformulation is fully developed in continuous time and in a deterministic setup. The effectiveness of the method has been shown on a benchmark simulation example.

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