Controller Design through Random Sampling: an Example*

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Summary. In this chapter, we present the *scenario approach*, an innovative technology for solving convex optimization problems with an infinite number of constraints. This technology relies on random sampling of constraints, and provides a powerful means for solving a variety of design problems in systems and control. Specifically, the virtues of this approach are here illustrated by focusing on optimal control design in presence of input saturation constraints.

1 Introduction

Many problems in systems and control can be formulated as optimization problems, often times of *convex* type, [1, 2]. Convexity is appealing since 'convex' - as opposed to 'non-convex' - means 'solvable' in many cases.

In practical problems, an often-encountered feature is that the environment is uncertain, i.e. some elements and/or variables are not known with precision. This leads naturally to *robust* convex optimization. Similarly, design against uncertain signals and/or disturbances gives rise to optimization of the robust type.

A robust convex optimization problem is expressed in mathematical terms as

$$\text{RCP}: \min_{x \in \mathbb{R}^n} g(x) \text{ subject to:} \qquad (1)$$
$$f_\delta(x) \leq 0, \ \forall \delta \in \Delta,$$

where δ is the uncertain parameter, and $g(x)$ and $f_\delta(x)$ are convex functions in the n-dimensional optimization variable x for every δ within the uncertainty

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set Δ. An example of formalization of a control problem as RCP is provided in the next section.

Often times, Δ is a set containing an infinite number of instances. If e.g. δ represents the uncertain gain of a plant and this gain is known to take on value in some interval, Δ is such an interval. In the example discussed in this chapter, Δ is the infinite set of possible disturbances entering a given system.

Problems with a finite number of optimization variables and an infinite number of constraints are called *semi-infinite* optimization problems in the mathematical programming literature. It is well known that these problems are difficult to solve and they have been proven NP-hard in many cases, [3, 4, 5, 6].

In [7, 8], an innovative technology called 'scenario approach' has been introduced to deal with semi-infinite convex programming at a very general level. The main thrust of this technology is that solvability can be obtained through random sampling of constraints provided that a probabilistic relaxation of the worst-case robust paradigm of (1) is accepted. When dealing with problems in systems and control, the scenario approach opens up new avenues for working out solutions in many different contexts.

The scenario approach is presented in this chapter in an easy-to-follow manner by way of an example in optimal control with input saturation constraints.

2 An optimal control problem with constraints

Consider the following control problem: given a linear system affected by a disturbance belonging to some class, design a feedback controller that attenuates the effect of the disturbance on the system output, while avoiding saturation of the control action due to actuator limitations.

Although quite standard in practice, this design problem is generally difficult to solve because of the presence of saturation constraints, and trial-and-error solutions are often adopted.

In this section, we illustrate a new approach to address this control problem in a systematic and optimal way. As we shall see, the proposed design methodology relies on the re-formulation of the problem as a robust convex optimization program by adopting an appropriate parametrization of the controller. Solvability of this robust convex optimization program is then attained through the scenario optimization technology.

2.1 Problem formulation

We consider a discrete time linear system with scalar input and scalar output, $u(t)$ and $y(t)$, governed by the following equation:

$$y(t) = G(z)u(t) + d(t), \tag{2}$$

where $G(z)$ is a stable transfer function and $d(t)$ is an additive disturbance.

Our objective is to determine a feedback control law

$$u(t) = C(z)y(t) \tag{3}$$

such that the disturbance $d(t)$ is optimally attenuated for every realization of $d(t)$ in some set of possible realizations $\mathcal{D}$, and such that the control input keeps within certain saturation limits. For example, $\mathcal{D}$ can be the set of step functions with specified maximum amplitude or the set of sinusoids with frequency in a certain range. A precise formalization of the optimization problem is next given.

Consider the finite-horizon 2-norm $\sum_{t=1}^{M} y(t)^2$ of the closed-loop system output. This norm quantifies the effect of the disturbance $d(t)$. For simplicity, we here consider (2) and (3) initially at rest, namely $G(z)u(t)$ represents an infinite backwards expansion $\sum_{j=1}^{\infty} g_j u(t-j)$ where $u(t-j) = 0$ for $t-j \leq 0$, and similarly for $C(z)y(t)$.

The goal is to minimize the worst-case disturbance effect

$$\max_{d(t)\in\mathcal{D}} \sum_{t=1}^{M} y(t)^2, \tag{4}$$

while maintaining the control input $u(t)$ within a saturation limit u_{bound}:

$$\max_{1\leq t\leq M} |u(t)| \leq u_{\text{bound}},\ \forall d(t) \in \mathcal{D}. \tag{5}$$

Controller $C(z)$ is expressed in terms of an Internal Model Control (IMC) parametrization, [9]:

$$C(z) = \frac{Q(z)}{1 + G(z)Q(z)}, \tag{6}$$

where $G(z)$ is the system transfer function and $Q(z)$ is a free-to-choose transfer function (see Figure 1).

Expression of $C(z)$ in (6) is totally generic, in that, given a $C(z)$, a $Q(z)$ can be always found generating that $C(z)$ through expression (6). The advantage of (6) is that the set of all controllers that closed-loop stabilize $G(z)$ is simply obtained from (6) by letting $Q(z)$ vary over the set of all stable transfer functions (see [9] for more details).

With (6) in place, the control input $u(t)$ and the controlled output $y(t)$ are given by:

$$u(t) = Q(z)d(t) \tag{7}$$

$$y(t) = [G(z)Q(z) + 1]d(t). \tag{8}$$

The distinctive feature of these expressions is that $u(t)$ and $y(t)$ are affine in $Q(z)$. Consequently, (4) is a convex cost in $Q(z)$ and (5) are convex constraints.

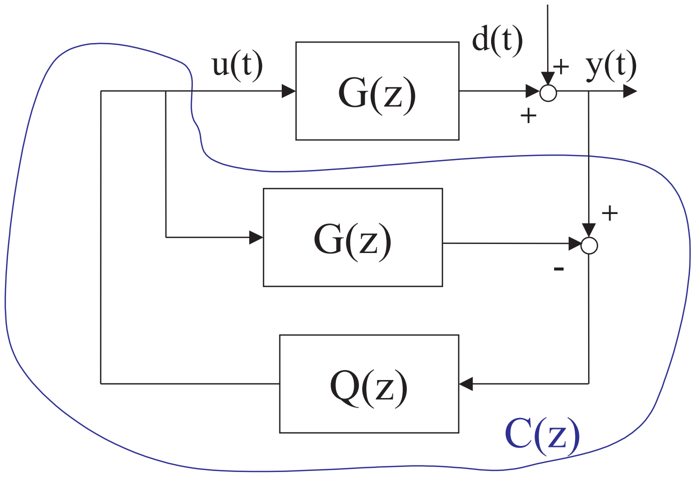

Fig. 1. The IMC parameterization.

In the sequel, we refer to the case where $Q(z)$ is selected from a family of stable transfer functions linearly parameterized in $\gamma := [\gamma_0\, \gamma_1\, \ldots, \gamma_k]^T \in \mathbb{R}^{k+1}$, i.e.

$$Q(z) = \gamma_0\beta_0(z) + \gamma_1\beta_1(z) + \gamma_2\beta_2(z) + \cdots + \gamma_k\beta_k(z), \tag{9}$$

where $\beta_i(z)$'s are pre-specified stable transfer functions. Note that linearity in γ is important because, due to convexity of (4) and (5) in $Q(z)$, it translates into convexity of the problem in γ.

A common choice for the $\beta_i(z)$'s functions is to set them equal to pure 'delays': $\beta_i(z) = z^{-i}$, leading to

$$Q(z) = \gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \cdots + \gamma_k z^{-k}.$$

Another possibility is to let $\beta_i(z)$'s be Laguerre polynomials, [10, 11].

The control design problem can now be precisely formulated as follows:

$$\min_{\gamma, h \in \mathbb{R}^{k+2}} h \quad \text{subject to:} \tag{10}$$

$$\sum_{t=1}^{M} y(t)^2 \leq h,\ \forall d(t) \in \mathcal{D}, \tag{11}$$

$$\max_{1 \leq t \leq M} |u(t)| \leq u_{\text{bound}},\ \forall d(t) \in \mathcal{D}. \tag{12}$$

Due to (11), h represents an upper bound to the output 2-norm $\sum_{t=1}^{M} y(t)^2$ for any realization of $d(t)$. Such an upper bound is minimized in (10) under the additional constraint (12) that $u(t)$ does not exceed the saturation limits.

2.2 Rewriting problem (10)–(12) in a more explicit form

By (7) and (8) and the parametrization of $Q(z)$ in (9), the input and the output of the controlled system can be expressed as

$$u(t) = \big(\gamma_o\beta_0(z) + \ldots + \gamma_k\beta_k(z)\big)d(t) \tag{13}$$

$$y(t) = G(z)\big(\gamma_o\beta_0(z) + \ldots + \gamma_k\beta_k(z)\big)d(t) + d(t). \tag{14}$$

Let us define the following vectors containing filtered versions of the disturbance $d(t)$:

$$\phi(t) := \begin{bmatrix} \beta_0(z)d(t) \\ \beta_1(z)d(t) \\ \vdots \\ \beta_k(z)d(t) \end{bmatrix} \quad \text{and} \quad \psi(t) := \begin{bmatrix} G(z)\beta_0(z)d(t) \\ G(z)\beta_1(z)d(t) \\ \vdots \\ G(z)\beta_k(z)d(t) \end{bmatrix}. \tag{15}$$

Then, (13) and (14) can be re-written as

$$u(t) = \phi(t)^T\gamma$$
$$y(t) = \psi(t)^T\gamma + d(t),$$

and $\sum_{t=1}^{M} y(t)^2 = \gamma^T A\gamma + B\gamma + C$, where

$$A = \sum_{t=1}^{M} \psi(t)\psi(t)^T \quad B = 2\sum_{t=1}^{M} d(t)\psi(t)^T \quad C = \sum_{t=1}^{M} d(t)^2 \tag{16}$$

are matrices that depend on $d(t)$ only.

With all these positions, (10)–(12) rewrites as

$$\min_{\gamma,h\in\mathbb{R}^{k+2}} h \text{ subject to:} \tag{17}$$
$$\gamma^T A\gamma + B\gamma + C \leq h, \ \forall d(t) \in \mathcal{D}$$
$$-u_{\text{bound}} \leq \phi(t)^T\gamma \leq u_{\text{bound}}, \ \forall t \in \{1, 2, \ldots, M\}, \ \forall d(t) \in \mathcal{D}.$$

Compared with the general form (1), the optimization variable x is here (γ, h) and has size $n = k+2$, and the uncertain parameter δ is the disturbance realization $d(t)$ taking value in the set $\Delta = \mathcal{D}$. Note that, given $d(t)$, quantities A, B, C, and $\phi(t)$ are fixed so that the first constraint in (17) is quadratic, while the others are linear.

Typically, the set $\mathcal{D}$ of disturbance realizations has infinite cardinality. Hence, problem (17) is a semi-infinite convex optimization problem.

2.3 Randomized solution through the scenario technology

As already pointed out in the introduction, semi-infinite convex optimization problems like (17) are difficult to solve. The idea of the scenario approach is that solvability can be recovered if some relaxation in the concept of solution is accepted. In the context of our control design problem, this means requiring that the constraints in (17) are satisfied for all disturbance realizations but a small fraction of them (*chance-constrained* approach).

The scenario approach goes as follows. Since we are unable to deal with the wealth of constraints in (17), we concentrate attention on just a few of them and extract at random N disturbance realizations $d(t)$ according to some probability distribution P introduced over $\mathcal{D}$. This probability distribution should reflect the likelihood of the different disturbance realizations. If no hint is available on which realization is more likely to occur, then the uniform distribution can be adopted. Only these extracted instances ('scenarios') are considered in the scenario optimization:

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extract N independent identically distributed realizations $d(t)_1$, $d(t)_2$, ..., $d(t)_N$ from $\mathcal{D}$ according to P. Then, solve the scenario convex program:

$$\mathrm{SCP}_N: \min_{\gamma, h \in \mathbb{R}^{k+2}} h \text{ subject to:} \tag{18}$$

$$\begin{aligned} &\gamma^T A_i \gamma + B_i \gamma + C_i \leq h, \ i = 1, \ldots, N, \\ &-u_{\text{bound}} \leq \phi(t)_i^T \gamma \leq u_{\text{bound}}, \ \forall t \in \{1, 2, \ldots, M\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, N, \end{aligned}$$

where A_i, B_i, C_i, and $\phi(t)_i$ are as in (16) and (15) for $d(t) = d(t)_i$.

Letting (γ_N^*, h_N^*) be the solution to SCP_N, γ_N^* returns the designed controller parameter.

The implementation of the scenario optimization requires that one picks N realizations of the disturbance and computes A_i, B_i, C_i, and $\phi(t)_i$ in correspondence of the extracted realizations. Since these quantities are artificially generated (that is they are not actual measurements coming from the system, but, instead, they are computer-generated), the proposed control design methodology can as well be seen as a *simulation-based approach.*

SCP_N is a standard convex optimization problem with a finite number of constraints, and therefore easily solvable. On the other hand, it is spontaneous to ask: what kind of solution is one provided by SCP_N? Specifically, what can we claim regarding the behavior of the designed control system for all other disturbance realizations, those we have not taken into consideration while solving the control design problem?

The above question is of the 'generalization' type in a learning-theoretic sense: we want to know whether and to what extent the solution generalizes

in constraints satisfaction, from seen constraints to unseen ones. Certainly, any generalization result calls for some structure as no generalization is possible if no structure linking what has been seen to what has not been seen is present. The formidable fact in the context of convex optimization is that - by underlying hidden links - the solution of SCP$_N$ always generalizes well, with no extra assumptions.
We have the following theorem (see Corollary 1 in [8]).

Theorem 1. *Select a 'violation parameter' $\epsilon \in (0,1)$ and a 'confidence parameter' $\beta \in (0,1)$. Let $n = k + 2$.*
If

$$N = \left\lceil \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n + \frac{2n}{\epsilon} \ln \frac{2}{\epsilon} \right\rceil \tag{19}$$

($\lceil \cdot \rceil$ denotes the smaller integer greater than or equal to the argument), then, with probability no smaller than $1-\beta$, the solution (γ_N^*, h_N^*) to (18) satisfies all constraints of problem (17) with the exception of those corresponding to a set of disturbance realizations whose probability is at most ϵ. $\square$

Let us read through the statement of this theorem in some detail. If we neglect the part associated with β, then, the result simply says that, by sampling a number of disturbance realizations as given by (19), the solution (γ_N^*, h_N^*) to (18) violates the constraints corresponding to other realizations with a probability that does not exceed a *user-chosen* level ϵ. This corresponds to say that – for other, unseen, $d(t)$'s – constraints (11) and (12) are violated with a probability at most ϵ. From (11) we therefore see that the found h_N^* provides an upper bound for the output 2-norm $\sum_{t=1}^{M} y(t)^2$ valid for any realizations of the disturbance with exclusion of at most an ϵ-probability set, while (12) guarantees that, with the same probability, the saturation limits are not exceeded.

As for the probability $1-\beta$, one should note that (γ_N^*, h_N^*) is a random quantity because it depends on the randomly extracted disturbance realizations. It may happen that the extracted realizations are not representative enough (one can even stumble on an extraction as bad as selecting N times the same realization!). In this case no generalization is certainly expected, and the portion of unseen realizations violated by (γ_N^*, h_N^*) is larger than ϵ. Parameter β controls the probability of extracting 'bad' realizations, and the final result that (γ_N^*, h_N^*) violates at most an ϵ-fraction of realizations holds with probability $1-\beta$.

In theory, β plays an important role and selecting $\beta = 0$ yields $N = \infty$. For any practical purpose, however, β has very marginal importance since it appears in (19) under the sign of logarithm: we can select β to be such a small number as 10^{-10} or even 10^{-20}, in practice zero, and still N does not grow significantly.

3 Numerical example

A simple example illustrates the controller design procedure.

With reference to (2), let

$$G(z) = \frac{0.2}{z - 0.8},$$

and let the additive output disturbance be a piecewise constant signal that varies from time to time, at a low rate, of an amount bounded by some given constant. Specifically, let the set of admissible realizations $\mathcal{D}$ consists of piecewise constant signals changing at most once over any time interval of length 50, and taking value in $[-1, 1]$.

As for the IMC parametrization $Q(z)$ in (9), we choose $k = 1$ and $Q(z) = \gamma_0 + \gamma_1 z^{-1}$.

A control design problem (10)–(12) is considered with $M = 300$, and for two different values of the saturation limit u_{bound}: 10 and 1. Probability P is implicitly assigned by the recursive equation

$$d(t+1) = \big(1 - \mu(t)\big)d(t) + \mu(t)v(t+1),$$

initialized with $d(1) = v(1)$, where $\mu(t)$ is a $\{0, 1\}$-valued process ($\mu(t) = 1$ at times where a jump occurs), and $v(t)$ is a sequence of i.i.d. random variables uniformly distributed in $[-1, 1]$ ($v(t)$ is the new $d(t)$ value). $\mu(t)$ is generated according to

$$\mu(t) = \alpha(t) \prod_{k=1}^{50} \big(1 - \mu(t-k)\big),$$

initialized with $\mu(0) = \mu(-1) = \cdots = \mu(-49) = 0$, where $\alpha(t)$ is a sequence of i.i.d. $\{0, 1\}$-valued random variables taking value 1 with probability 0.01. An admissible realization of $d(t)$ in $\mathcal{D}$ is reported in Figure 2.

In the scenario approach we let $\epsilon = 5 \cdot 10^{-2}$ and $\beta = 10^{-10}$. Correspondingly, N given by (19) is $N = 1370$.

From Theorem 1, with probability no smaller than $1 - 10^{-10}$, the obtained controller achieves the minimum of $\sum_{t=1}^{M} y(t)^2$ over all disturbance realizations, except a fraction of them of size smaller than or equal to 5%. At the same time, the control input $u(t)$ is guaranteed not to exceed the saturation limit u_{bound} except for the same fraction of disturbance realizations.

3.1 Simulation results

For $u_{\mathrm{bound}} = 10$, we obtained $Q(z) = -4.9931 + 4.0241 z^{-1}$ and, correspondingly, the transfer function $F(z) = 1 + Q(z)G(z)$ between $d(t)$ and $y(t)$ (closed-loop sensitivity function) was

$$F(z) = 1 + (-4.993 + 4.024 z^{-1}) \frac{0.2}{z - 0.8} \simeq 1 - z^{-1}.$$

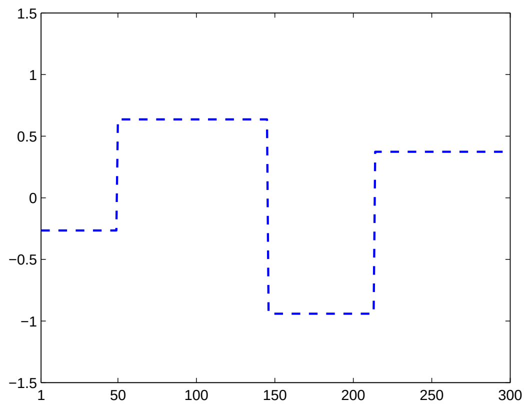

Fig. 2. A disturbance realization.

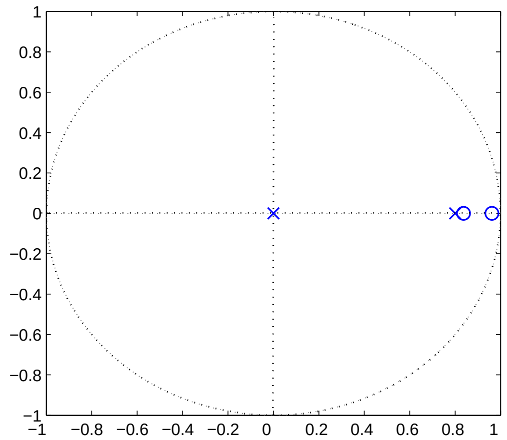

Fig. 3. Pole-zero plot of $F(z)$ when $u_{\text{bound}} = 10$. The poles are plotted as x's and the zeros are plotted as o's.

The pole-zero plot of $F(z)$ is in Figure 3.

Since $y(t) = F(z)d(t) \simeq d(t) - d(t-1)$, then, when $d(t)$ has a step variation, $y(t)$ changes of the same amount and, when the disturbance gets constant, $y(t)$ is immediately brought back to zero and maintained equal to zero until the next step variation in $d(t)$ (see Figure 4). The obtained solution that $F(z)$ is approximately a FIR (Finite Impulse Response) of order 1 with zero DC-gain is not surprising considering that $d(t)$ varies at a low rate.

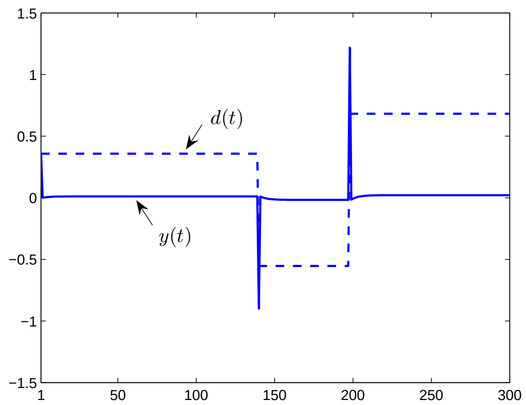

Fig. 4. Disturbance realization and corresponding output of the controlled system for $u_{\mathrm{bound}} = 10$.

In the controller design just described, the limit $u_{\mathrm{bound}} = 10$ played no role in that constraints $-u_{\mathrm{bound}} \leq \phi(t)_i^T \gamma \leq u_{\mathrm{bound}}$ in problem (18) were not active at the found solution. As u_{bound} is decreased, the saturation limits become more stringent and affect the solution.

For $u_{\mathrm{bound}} = 1$, the following scenario solution was found $Q(z) = -0.991 + 0.011z^{-1}$, which corresponds to the sensitivity function:

$$F(z) = 1 + (-0.991 + 0.011z^{-1})\frac{0.2}{z - 0.8} \simeq \frac{z - 0.9960}{z - 0.8}.$$

The pole-zero plot of $F(z)$ is in Figure 5, while Figure 6 represents $y(t)$ obtained through this new controller for the same disturbance realization as in Figure 4. Note that the time required to bring $y(t)$ back to zero after a disturbance jump is now longer than 1 time unit, owing to saturation constraints on $u(t)$.

The optimal control cost value h_N^* is $h_N^* = 9.4564$ for $u_{\mathrm{bound}} = 10$ and $h_N^* = 27.4912$ for $u_{\mathrm{bound}} = 1$. As expected, the control cost increases as u_{bound} becomes more stringent.

The numerical example of this section is just one instance of application of the scenario approach to controller selection. The introduced methodology is of general applicability to diverse situations with constraints of different type, presence of reference signals, etc.

4 Conclusions: a final glance over the scenario world

In this chapter, we considered an optimal disturbance rejection problem with limitations on the control action and showed how it can be effectively ad-

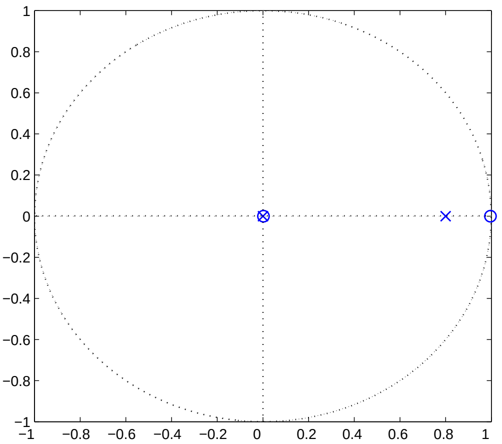

Fig. 5. Pole-zero plot of $F(z)$ when $u_{\text{bound}} = 1$. The poles are plotted as x's and the zeros are plotted as o's.

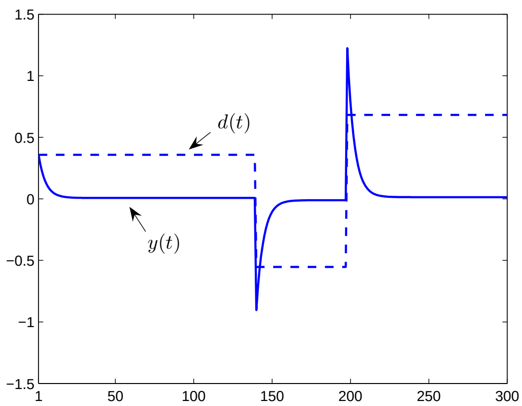

Fig. 6. Disturbance realization and corresponding output of the controlled system for $u_{\text{bound}} = 1$.

dressed by means of the so-called scenario technology. This approach basically consists of the following main steps:

- reformulation of the problem as a robust (usually with infinite constraints) *convex* optimization problem;
- randomization over constraints and resolution (by means of standard numerical methods) of the so obtained *finite* optimization problem;

- evaluation of the constraint satisfaction level of the obtained solution through Theorem 1.

The applicability of the scenario methodology is certainly not limited to optimal disturbance rejection problems and, indeed, this same methodology has been applied to a number of different endeavors in systems and control.

Robust control, for example, is a natural setting for the scenario approach, since robust control performance requirements can be often translated into optimization with an infinite number of constraints. The reader is referred to [12, 13, 8], where the scenario methodology has been applied to robust stabilization, LPV (Linear Parameter Varying) control, and robust pole assignment.

Another setting in which the scenario approach proved powerful is the identification of interval predictor models (i.e. models returning a prediction interval instead of a single prediction value), [14, 15]. Here, constraints are given by observed data and optimization is performed to shrink the interval model as tightly as possible around data.

Finally, the scenario approach is currently being applied to system identification through an innovative min-max perspective.

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