SIAM J. CONTROL OPTIM.
Vol. 39, No. 1, pp. 133–140

PERSISTENCE OF EXCITATION PROPERTIES FOR TIME-VARYING AUTOREGRESSIVE SYSTEMS*

SERGIO BITTANTI† AND M. C. CAMPI‡

Abstract. It is well known that a crucial property for the effective identification of time-varying systems is that the data carry continual information on the parameters to be estimated. As a matter of fact, only in this case can the identification algorithm rely on fresh information in forming a reliable estimate of the current value of these parameters. This concept has been formalized in the system identification literature under the name of *persistence of excitation*.

In this paper, the persistence of excitation property is studied for a class of time-varying systems (that includes the standard autoregressive model as a particular case) and conditions for it to hold are derived.

Key words. time-varying models, persistence of excitation, autoregressive models, system identification

AMS subject classifications. Primary, 93E12; Secondary, 93B30

PII. S0363012998343483

1. Introduction.

In the last two decades, a considerable effort has been put into the comprehension of identification methods for the estimation of time-varying systems.

A huge stream of research has been devoted to situations that somehow reduce to the problem of estimating constant unknown parameters. This is, for instance, the case of the so-called random coefficient autoregressive models; see, e.g., Nicholls and Quinn (1982), Chow (1983), and Beran and Hall (1992). These models are characterized by parameters which are randomly fluctuating according to the law $\vartheta(t) = \bar{\vartheta} + \delta(t)$, $\delta(t)$ being an independent sequence. In this framework the main concern is the consistent estimation of the mean value $\bar{\vartheta}$. Another kind of time-varying systems which has attracted interest in recent years are the so-called nearly nonstationary autoregressive models. In this case, the time-varying parameters are asymptotically convergent and the corresponding asymptotic invariant model exhibits singularities on the unit circle. The limiting distribution of the estimation error when the identification is performed via the standard least squares algorithm is studied, e.g., in Cox and Llatas (1991); see also Cox (1991).

In the above literature, the fact that the estimated parameters are in fact constant makes the estimation task simpler than in truly time-varying situations. As a matter of fact, when the parameters are constant, the same unknowns are estimated through time and it is expected that a consistent estimate can be formed under the sole condition that data carry enough information in the long run. On the other hand, when the goal is that of estimating truly time-varying parameters, one has to somehow guarantee that a certain amount of information is available over any finite interval of time. As a matter of fact, only in this way can the identification algorithm rely on fresh information in forming a reliable estimate of the current value of the parameters.

*Received by the editors August 12, 1998; accepted for publication (in revised form) February 16, 2000; published electronically August 17, 2000.

http://www.siam.org/journals/sicon/39-1/34348.html

†Dipartimento di Elettronica e Informazione, Politecnico di Milano, piazza Leonardo da Vinci 32, 20133 Milano, Italy (bittanti@elet.polimi.it).

‡Dipartimento di Elettronica per l'Automazione, Universita' di Brescia, via Branze 38, 25123 Brescia, Italy (campi@ing.unibs.it).

This idea is well known in the identification literature under the name of *persistence of excitation*.

Letting $\varphi(\cdot)$ be the observation vector, a persistence of excitation condition which has been widely used in the literature takes the form

$$\text{pr}\left(\lambda_{\min}\left\{\sum_{i=t}^{t+s-1}\frac{\varphi(i)\varphi(i)'}{1+\|\varphi(i)\|^2}\right\}\geq k_1 \,\middle|\, \sigma_{t-1}\right)\geq k_2 \qquad \forall t, \tag{1}$$

where $\lambda_{\min}$ denotes the minimum eigenvalue and σ_t is the so-called σ-algebra of the past, that is, the σ-algebra generated by all system processes up to time t. Roughly, this condition requires that, whatever the past evolution of the system might have been, the information carried by data over the next s time points spans the entire parameter space with a finite nonzero probability.

Condition (1) was first introduced in Guo (1990) in a form that is slightly different from but equivalent to (1), and has henceforth been used in many different contributions.

Under (1), Guo (1990) proves stability and convergence results for a Kalman filter based algorithm used in the estimation of time-varying parameters generated by a random-walk–type equation. In the paper of Bittanti and Campi (1994) it is proven that a forgetting factor least squares identification algorithm provides bounded estimates if condition (1) is met and the forgetting factor is chosen to be larger than a certain threshold. Another contribution using the persistence of excitation condition (1) is Campi (1994). There, an explicit expression for the asymptotic estimation error is given for a forgetting factor based least squares algorithm. This bound shows the dependence of the estimation error on the speed of the time variability of the parameters and the variance of noise.

There are many more contributions on system identification where significant properties are proven under conditions related to (1). Among others, we cite Bittanti and Campi (1991a, 1991b); Guo, Ljung, and Priouret (1993); Guo and Ljung (1995a, 1995b); and Campi (1997). An additional interesting paper is Ravikanth and Meyn (1999), where a lower bound for the estimation error valid for any identification algorithm is worked out.

In all of the above-mentioned contributions, condition (1) is taken for granted or proven only in certain specific situations. In the present paper we address the problem of verifying that such a condition is in fact satisfied for a class of time-varying systems which includes, but is not limited to, autoregressive systems. In this way, all the results proven in these contributions can in fact be applied to this class of models.

The paper is organized as follows. In section 2 the system class is introduced. The persistence of excitation condition is then discussed in section 3.

2. The system. Let us consider a time-varying state variable system described by the equation

$$\varphi(t) = G(t)\varphi(t-1) + v(t). \tag{2}$$

In (2), $\varphi(t) \in \mathbb{R}^n$ is the so-called observation vector and it is a measurable signal, and $v(t)$ is a remote unmeasurable noise that plays the role of a latent variable in the generation of $\varphi(t)$. Throughout the paper, it is assumed that matrices $G(t)$ form a strictly stationary stochastic process.

The transition matrix associated with $G(t)$ is defined as

$$\Phi(t, s) := G(s)G(s+1)\cdots G(t).$$

A typical example of system (2) is a time-varying scalar autoregressive model of the form

$$y(t) = a_1(t)y(t-1) + a_2(t)y(t-2) + \cdots + a_n(t)y(t-n) + d(t). \tag{3}$$

In this case, by letting

$$G(t) = \begin{bmatrix} a_1(t) & a_2(t) & \cdots & a_n(t) \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \quad \varphi(t) = \begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-n+1) \end{bmatrix}, \quad v(t) = \begin{bmatrix} d(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

system (2) is immediately recovered. Clearly, system (2) can accommodate many other specific situations than the autoregressive system (3).

The following assumptions are made on system (2).

Assumption 1. $v(\cdot)$ is a zero-mean, bounded independently and identically distributed (i.i.d.) sequence, independent of $G(\cdot)$.

Assumption 2. $\exists \rho$: $\rho^{-t}\|\Phi(t,0)\| \leq \alpha$ $\forall t$ almost surely.

Clearly, Assumption 2 is an exponential stability condition. It is worthwhile pointing out that there is a milder stability condition that could be considered.

Assumption 2′. $\exists \rho$: $\limsup_{t\to\infty} \rho^{-t}\|\Phi(t,0)\| = 0$ almost surely.

Assumption 2′ is a stability assumption of stochastic type that requires $\|\Phi(t,0)\|$ to go to zero exponentially fast with *asymptotic* deterministic rate ρ. On the other hand, Assumption 2 imposes restrictions for any *finite* t. It is in fact a truly deterministic stability assumption.

It is easy to see that Assumption 2′ is equivalent to

$$\limsup_{t\to\infty} t^{-1} \log \|\Phi(t,0)\| \leq -\gamma < 0 \quad \text{almost surely.} \tag{4}$$

This last condition has been discussed (in a continuous-time setting) by Solo (1994). Among other things, Solo provides conditions on the eigenvalues of the stochastic matrix $G(t)$ such that (4) holds true.

Finally, notice that, since $G(\cdot)$ is strictly stationary, Assumption 2 is equivalent to

$$\|\Phi(t,s)\| \leq \alpha\rho^{t-s} \quad \forall t, s \qquad \text{almost surely.}$$

3. Main result: Persistence of excitation condition. In this section, the persistence of excitation condition is discussed and necessary conditions for it to hold are derived.

For subsequent use, we introduce the σ-algebra generated by the past of $v(\cdot)$ and the past, present, and future of $G(\cdot)$:

$$\zeta_t = \sigma(v(i), \quad i \leq t; \quad G(\cdot)).$$

Notice that $\varphi(t)$ is measurable with respect to ζ_t.

For the sake of clarity, we point out that the σ-algebra of the past in condition (1) is given by

$$\sigma_t = \sigma(v(j), G(j), j \leq t).$$

We start by proving the following proposition which is a law of large numbers of conditional type for system (2).

PROPOSITION 3.1. *Under Assumptions* 1 *and* 2,

$$E\left[\left\| \frac{1}{k} \sum_{i=t}^{t+k-1} (\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t]) \right\|^2 \middle| \zeta_t \right] \longrightarrow 0 \quad \text{as } k \to \infty,$$

uniformly with respect to both time t and probability outcome.

Proof. The following chain of inequalities holds true:

$$\begin{aligned} &E\left[\left\| \frac{1}{k} \sum_{i=t}^{t+k-1} (\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t]) \right\|^2 \middle| \zeta_t \right] \\ &\quad \leq \frac{1}{k^2} n \left\| E\left[\sum_{i,j=t}^{t+k-1} (\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t]) \right.\right. \\ &\qquad\qquad \left.\left. \times \; (\varphi(j)\varphi(j)' - E[\varphi(j)\varphi(j)' \mid \zeta_t]) \middle| \zeta_t \right] \right\| \end{aligned}$$

(since, for any stochastic matrix $M \geq 0$ of dimension n, $E[\|M\|] \leq n\|E[M]\|$)

$$\begin{aligned} &\leq \frac{1}{k^2} n \sum_{i,j=t}^{t+k-1} \|E[(\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t]) \\ &\qquad\qquad \times (\varphi(j)\varphi(j)' - E[\varphi(j)\varphi(j)' \mid \zeta_t]) \mid \zeta_t]\| \\ &\leq \frac{1}{k^2} 2n \sum_{\substack{i,j=t \\ j \geq i}}^{t+k-1} \|E[(\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t]) \\ &\qquad\qquad \times (E[\varphi(j)\varphi(j)' \mid \zeta_i] - E[\varphi(j)\varphi(j)' \mid \zeta_t]) \mid \zeta_t]\|. \end{aligned}$$

In this last expression, the norm of $(\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t])$ is deterministically bounded in view of the boundedness of $v(\cdot)$ (Assumption 1) and the exponential stability of the system (Assumption 2).

Therefore, to complete the proof it suffices to prove that the norm of $(E[\varphi(j)\varphi(j)' \mid \zeta_i] - E[\varphi(j)\varphi(j)' \mid \zeta_t])$, $j \geq i \geq t$, is bounded by a deterministic function of $j - i$ only, which tends exponentially to zero as $j - i \to \infty$.

Set $\varphi(r \mid s) := E[\varphi(r) \mid \zeta_s]$, $r \geq s$. Since $v(\cdot)$ is an independent sequence, we have

$$\varphi(r \mid s) = \sum_{k=-\infty}^{s+1} \Phi(r,k) v(k-1).$$

Taking into account the exponential stability assumption (Assumption 2) and that the noise $v(\cdot)$ is bounded (Assumption 1), this last expression shows that $\|\varphi(r \mid s)\|$ is bounded by a deterministic function of $r-s$ only, which tends exponentially to zero as $r-s\to\infty$. The term $(E[\varphi(j)\varphi(j)' \mid \zeta_i] - E[\varphi(j)\varphi(j)' \mid \zeta_t])$ can now be handled as follows:

$$\begin{aligned} &E[\varphi(j)\varphi(j)' \mid \zeta_i] - E[\varphi(j)\varphi(j)' \mid \zeta_t] \\ &\quad = E[(\varphi(j \mid i) + (\varphi(j) - \varphi(j \mid i)))(\varphi(j \mid i) + (\varphi(j) - \varphi(j \mid i)))' \mid \zeta_i] \\ &\qquad - E[(\varphi(j \mid t) + (\varphi(j) - \varphi(j \mid t)))(\varphi(j \mid t) + (\varphi(j) - \varphi(j \mid t)))' \mid \zeta_t] \\ &\quad = \varphi(j \mid i)\varphi(j \mid i)' - \varphi(j \mid t)\varphi(j \mid t)' - \sum_{k=t+2}^{i+1} \Phi(j,k)\Delta\phi(j,k)', \end{aligned}$$

where $\Delta := E[v(t)v(t)']$. The thesis follows by observing that the norm of each of these three terms is bounded by a deterministic function of $j-i$ only, which tends exponentially to zero as $j-i\to\infty$. □

Notice that, up to now, no conditions have been introduced guaranteeing that vector $\varphi(\cdot)$ is somehow exciting (in fact, under Assumptions 1 and 2, $v(\cdot)$ and/or $G(\cdot)$ may well be identically zero). We now introduce an extra condition (Assumption 3 below) which can be interpreted as an excitation condition. We anticipate that, in view of Proposition 1, Assumption 3 immediately leads to concluding that $\varphi(\cdot)$ is persistently exciting in the sense of definition (1) (see Theorem 1 below). The fact that Assumption 3 holds true in many situations of interest (e.g., for the autoregressive system (2)) is discussed immediately after the theorem.

Assumption 3. $E[\varphi(i)\varphi(i)' \mid \zeta_t] \geq H > 0\ \forall i \geq t+\bar{n}$, for some integer $\bar{n}$.

THEOREM 3.2. *Under Assumptions* 1–3*, there exist an integer s and two positive real numbers k_1 and k_2 such that the persistence of excitation condition* (1) *is satisfied.*

Proof. Recalling that, for any pair of positive semidefinite matrices C and D, $\lambda_{\min}[C] \geq \lambda_{\min}[D] - \|C-D\|$, one obtains

$$\begin{aligned} \lambda_{\min}\left\{\frac{1}{k}\sum_{i=t}^{t+k-1}\varphi(i)\varphi(i)'\right\} &\geq \lambda_{\min}\left\{\frac{1}{k}\sum_{i=t}^{t+k-1}E[\varphi(i)\varphi(i)' \mid \zeta_t]\right\} \\ &\quad - \left\|\frac{1}{k}\sum_{i=t}^{t+k-1}(\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)' \mid \zeta_t])\right\|. \end{aligned}$$

Take now conditional expectation of this last equation with respect to ζ_t. Thanks to Assumption 3 and Proposition 1, it is then apparent that there exist an integer s and a real number β such that

$$E\left[\lambda_{\min}\left\{\frac{1}{s}\sum_{i=t}^{t+s-1}\varphi(i)\varphi(i)'\right\}\middle|\zeta_t\right] \geq \beta > 0 \quad \forall t.$$

Then in view of the boundedness of $\varphi(\cdot)$ (Assumptions 1 and 2), we can conclude that there exist two positive real numbers k_1 and k_2 such that

$$\mathrm{pr}\left(\lambda_{\min}\left\{\sum_{i=t}^{t+s-1}\frac{\varphi(i)\varphi(i)'}{1+\|\varphi(i)\|^2}\right\} \geq k_1 \mid \zeta_t\right) \geq k_2 \quad \forall t.$$

Since the σ-algebra generated by $v(j)$ and $G(j)$, $j \leq t-1$, is coarser than ζ_t, the thesis follows. □

Next, we show that Assumption 3 holds true in the case of the autoregressive system (2). Take $\bar{n} = n$. Recalling that $\varphi(r \mid s) = E[\varphi(r) \mid \zeta_s]$, for any $j \in [i-n, i-1]$ one has

$$\begin{aligned} &E[\varphi(i)\varphi(i)' \mid \zeta_t] \\ &\quad = E[E[(\varphi(i \mid j) + (\varphi(i) - \varphi(i \mid j)))(\varphi(i \mid j) + (\varphi(i) - \varphi(i \mid j)))' \mid \zeta_j] \mid \zeta_t] \\ &\qquad \text{(since } j \geq t\text{)} \\ &\quad \geq E[(\varphi(i) - \varphi(i \mid j))(\varphi(i) - \varphi(i \mid j))' \mid \zeta_t]. \end{aligned}$$

Since $\varphi(i) - \varphi(i \mid j) = \sum_{k=j+2}^{i+1} \Phi(i,k)v(k-1)$, we have ($\sigma^2 := E[d(t)^2]$)

- for $j = i - 1$,

$$E[\varphi(i)\varphi(i)' \mid \zeta_t] \geq \operatorname{diag}(\sigma^2, 0, \ldots, 0) = \begin{bmatrix} \sigma^2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix};$$

- for $j = i - 2$,

$$\begin{aligned} E[\varphi(i)\varphi(i)' \mid \zeta_t] &\geq \Phi(i,i)\operatorname{diag}(\sigma^2, 0, \ldots, 0)\phi(i,i)' \\ &= \begin{bmatrix} \star & \star & 0 & \ldots & 0 \\ \star & \sigma^2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}; \end{aligned}$$

$(\cdots)$

- for $j = i - n$,

$$\begin{aligned} E[\varphi(i)\varphi(i)' \mid \zeta_t] &\geq \Phi(i, i-n+2)\operatorname{diag}(\sigma^2, 0, \ldots, 0)\Phi(i, i-n+2)' \\ &= \begin{bmatrix} \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \\ \star & \star & \star & \ldots & \star \\ \vdots & \vdots & \vdots & & \vdots \\ \star & \star & \star & \ldots & \sigma^2 \end{bmatrix}, \end{aligned}$$

where the $\star$'s are random entries, whose value is bounded uniformly with respect to time t and probability outcome. From the above relations, Assumption 3 easily follows with $\bar{n} = n$.

It is interesting to note that Assumption 3 holds in many extra situations. As a simple example, if $G(\cdot)$ is deterministic such that Assumption 2 holds, then Assumption 3 is met provided that the very minimal condition $E[\varphi(i)\varphi(i)'] > 0$ is satisfied.

The analysis has been conducted so far under the stability assumption, Assumption 2. It is, of course, of interest to investigate whether the persistence of excitation condition (1) still holds under the weaker stability assumption, Assumption 2$'$. Unfortunately, this is not the case, as the following simple example shows.

Example. Suppose that $\varphi(t)$ has two components and let $G(t) = \operatorname{diag}(g(t), 0)$. $g(\cdot)$ is an i.i.d. sequence such that $g(t) = 2$ with probability 0.5 and $g(t) = 0.25$ with probability 0.5. Finally, $v(t) = [v_1(t)v_2(t)]'$, where $v_1(\cdot)$ and $v_2(\cdot)$ are i.i.d. sequences independent of each other and of $g(\cdot)$ and take on values -1 and $+1$ with probability 0.5.

Assumptions 1 and 3 are trivially satisfied in this case. Assumption $2'$ is also satisfied. This is seen as follows. Since $\|\Phi(t,0)\| = g(0)g(1)\cdots g(t)$, for any given $\rho \in (0.1)$, we have

$$\frac{1}{t}\log(\rho^{-t}\|\Phi(t,0)\|) = \log\frac{1}{\rho} + \frac{1}{t}\sum_{s=0}^{t}\log g(s).$$

The second term in the left-hand side tends almost surely to $E[\log g(t)] = 0.5[\log 2 + \log 0.25] = -0.5\log 2$. Then, by taking ρ to be a real number such that $\log\frac{1}{\rho} - 0.5\log 2 < 0$, we have that $\frac{1}{t}\log(\rho^{-t}\|\Phi(t,0)\|)$ tends almost surely to a negative number. From this, we conclude that $\limsup_{t\to\infty}\rho^{-t}\|\Phi(t,0)\| = 0$ almost surely, that is, Assumption $2'$.

Next, we show that the persistence of excitation condition (1) is not satisfied in this case.

Given any real number h, let $\mathcal{A}_h := \{|\varphi_i(0)| > h\}$, where $\varphi_1(0)$ is the first component of $\varphi(0)$. Since $g(t)$ takes on value 2 with probability 0.5, $\varphi(0)$ has an unbounded distribution and so $\mathrm{pr}(\mathcal{A}_h) \neq 0 \quad \forall h$. Moreover, note that if $|\varphi_1(0)| > h$, then $|\varphi_1(t)| > (0.25)^t h - 5/4$. ($g(t)$ is either 2 or 0.25 and $|v_1(t)| = 1$.) Now, suppose by contradiction that (1) holds for certain fixed k_1, k_2, and s. Since $\varphi_2(i) = v_2(i)$ keeps bounded and $\|\varphi(i)\| \geq (0.25)^s h - 5/4 (i \in [1,s])$, a real number h exists such that condition $\lambda_{\min}\{\sum_{i=1}^{s}\frac{\varphi(i)\varphi(i)'}{1+\|\varphi(i)\|^2}\} \geq k_1$ is not satisfied on $\mathcal{A}_h$. So

$$E\left[\left\{\lambda_{\min}\left\{\sum_{i=1}^{s}\frac{\varphi(i)\varphi(i)'}{1+\|\varphi(i)\|^2}\right\} \geq k_1\right\}\cdot 1(\mathcal{A}_h)\right] = 0 < k_2 \cdot 1(\mathcal{A}_h)$$

(where $1(\mathcal{A}_h)$ is the indicator function of set $\mathcal{A}_h$) and this contradicts condition (1).

The above example shows that the persistence of excitation condition (1) does not hold under Assumption $2'$. On the other hand, almost all results in the identification literature (like those in Guo (1990), Bittanti and Campi (1994), or Campi (1994)) have been worked out under this condition (1). Consequently, at the present state of the art, it is not clear how to handle situations where the system is only characterized by a mild stability condition like Assumption $2'$. The above observation raises an interesting conceptual question: one may ask if it is possible to work out a persistence of excitation condition milder than (1), that holds true under Assumption $2'$ and still permits one to prove boundedness results for the identification algorithms. This issue is certainly worthy of further investigation.

REFERENCES

R. Beran and P. Hall (1992), *Estimating coefficient distributions in random coefficient regressions*, Ann. Statist., 20, pp. 1970–1984.

S. Bittanti and M.C. Campi (1991a), *Adaptive RLS algorithms under stochastic excitation—L^2 convergence analysis*, IEEE Trans. Automat. Control, 36, pp. 963–967.

S. Bittanti and M.C. Campi (1991b), *Adaptive RLS algorithms under stochastic excitation—strong consistency analysis*, Systems Control Lett., 17, pp. 3–8.

S. Bittanti and M.C. Campi (1994), *Bounded error identification of time-varying parameters by RLS techniques*, IEEE Trans. Automat. Control, 39, pp. 1106–1110.

M.C. Campi (1994), *Exponentially weighted least squares identification of time-varying systems with white disturbances*, IEEE Trans. Signal Process., 42, pp. 2906–2914.

M.C. Campi (1997), *Performance of RLS identification algorithms with forgetting factor: A* Φ*-mixing approach*, J. Math. Systems Estim. Control, 7, pp. 29–53.

G.C. Chow (1983), *Random and changing coefficient models*, in Handbook of Econometrics, Z. Griliches and M.D. Intriligator, eds., Handbooks in Econom. 2, North-Holland, Amsterdam, pp. 1213–1245.

D.D. Cox (1991), *Gaussian likelihood estimation for nearly nonstationary AR* (1) *processes*, Ann. Statist., 19, pp. 1129–1142.

D.D. Cox and I. Llatas (1991), *Maximum likelihood type estimation for nearly nonstationary autoregressive time series*, Ann. Statist., 19, pp. 1109–1128.

L. Guo (1990), *Estimating time-varying parameters by the Kalman filter based algorithm: Stability and convergence*, IEEE Trans. Automat. Control, 35, pp. 141–147.

L. Guo and L. Ljung (1995a), *Exponential stability of general tracking algorithms*, IEEE Trans. Automat. Control, 39, pp. 1376–1387.

L. Guo and L. Ljung (1995b), *Performance analysis of general tracking algorithms*, IEEE Trans. Automat. Control, 39, pp. 1388–1401.

L. Guo, L. Ljung, and P. Priouret (1993), *Performance analysis of the forgetting factor RLS algorithm*, Internat J. Adapt. Control Signal Process., 7, pp. 525–537.

D.F. Nicholls and B.G. Quinn (1982), *Random Coefficient Autoregressive Models: An Introduction*, Springer-Verlag, New York.

R. Ravikanth and S.P. Meyn (1999), *Bounds on achievable performance in the identification and adaptive control of time-varying systems*, IEEE Trans. Automat. Control, 44, pp. 670–682.

V. Solo (1994), *On the stability of slowly time-varying linear systems*, Math. Control Signals Systems, 7, pp. 331–350.