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Model quality assessment for Instrumental Variable methods: use of the asymptotic theory in practice

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***Abstract*— In this paper the problem of computing uncertainty regions for models identified through an Instrumental Variable technique is considered. Recently, it has been pointed out that, in certain operating conditions, the asymptotic theory of system identification (the most widely used method for model quality assessment) may deliver unreliable confidence regions. The aim of this paper is to show that, in an Instrumental Variable setting, the asymptotic theory exhibits a certain "robustness" that makes it reliable even when used with moderate data samples. Reasons for this are highlighted in the paper through a theoretical analysis and simulation examples.**

I. INTRODUCTION

Model quality assessment is a very important (and also challenging) problem in system identification. In fact, it has been widely recognized that an identified model is of little use in practical applications if an estimate of its reliability is not given along with the model itself. In other words, if S is the data-generating system and $\widehat{S}$ is the identified model, it is fundamental to characterize the system-model mismatch, i.e. the distance between S and $\widehat{S}$ (see [10], [8], [5] and [1]).

One of the best-known tools for model quality assessment is the asymptotic theory of system identification ([9] and [12]). The asymptotic theory works in a probabilistic framework and returns ellipsoidal confidence regions for S – namely, regions in the parameter space to which the data-generating system parameter belongs with a pre-assigned probability. It has been proved that such ellipsoidal confidence regions are asymptotically correct as the number of data points increases.

The major drawback with the use of the asymptotic theory is that, in real applications, only a finite number of data points is available. Therefore, the asymptotic theory holds true in practice only approximately, and it is a common experience that it returns sensible results in many cases, but not always. As a matter of fact, it has been recently shown that – in condition of *poor excitation* and depending on the underlying identification setting – the computed ellipsoid may as well be completely unreliable as an approximate confidence region (see [4] and [2]).

This limitation of the asymptotic theory can be quite severe because lack of excitation is common in many applications, particularly when the identification has to be performed in closed-loop with restricted bandwidth. This happens, for example, at the first iterations of iterative controller design schemes (see [3], [6], [13] and [7]). At a more general level, one can argue that model quality assessment becomes important when the system-model mismatch is significant and this occurs when the system is poorly excited, so that a good model quality assessment method should work properly especially in this case.

This paper focuses on an Instrumental Variable (*IV*) identification setting. The aim is to show that in this setting the asymptotic theory exhibits a certain "robustness" so that it can be safely used even in the case of poor excitation and for moderate data samples.

Structure of the paper

In Section II the *IV* identification setting is presented and a brief summary of the standard asymptotic theory is given. Moreover, the problems that may arise when using the asymptotic theory in presence of poor excitation are pointed out. Section III delivers a new asymptotic result, also valid in "singular" conditions, precisely defined in Section III. This result makes it possible to show in Section IV that the asymptotic theory for *IV* methods can be safely used even when data are poorly exciting. Some simulation results are given in Section V.

II. MODEL QUALITY ASSESSMENT FOR *IV* IDENTIFICATION

A. Mathematical setting

Throughout the paper we suppose that the data are generated by the following dynamical system, which is assumed to be asymptotically stable:

$$y(t) = \varphi(t)'\vartheta^o + v(t) \tag{1}$$

where

$$\varphi(t) = [y(t-1)\ \ldots\ y(t-n_a) \quad u(t-1)\ \ldots\ u(t-n_b)]'$$

is the n-vector ($n = n_a + n_b$) of observations and

$$\vartheta^o = [-a_1^o\ \ldots\ -a_{n_a}^o \quad b_1^o\ \ldots\ b_{n_b}^o]'$$

is the true system parameter n-vector.

The input $u(t)$ and the noise process $v(t)$ are generated according to the following scheme which encompasses closed-loop as well as open-loop configurations:

$$\begin{aligned} u(t) &= G(z^{-1})r(t) + H(z^{-1})e(t) \\ v(t) &= V(z^{-1})e(t), \end{aligned}$$

where $G(z^{-1})$, $H(z^{-1})$, $V(z^{-1})$, $r(t)$ and $e(t)$ satisfy the following assumption.

Assumption 1: The transfer functions $G(z^{-1})$, $H(z^{-1})$ and $V(z^{-1})$ are rational, proper and asymptotically stable. In addition, $V(z^{-1})$ has no zeroes on the unit circle in the complex plane. $e(t)$ is a sequence of independent zero mean random variables with variance $\lambda^2 > 0$ and such that $\mathbb{E}[|e(t)|^{4+\delta}] < \infty$, for some $\delta > 0$. $r(t)$ is a wide sense stationary, stochastic, ergodic, external input sequence. $r(t)$ and $e(t)$ are independent. $\square$

It is important to note that both $u(t)$ and $y(t)$ can be seen as the sum of two independent processes, one depending on $r(t)$ and the other depending on $e(t)$. That is, $u(t) = u_r(t) + u_e(t)$ and $y(t) = y_r(t) + y_e(t)$, where

$$\begin{aligned} u_r(t) &= G(z^{-1})r(t), && (2) \\ u_e(t) &= H(z^{-1})e(t), \\ y_r(t) &= \frac{B(z^{-1})}{A(z^{-1})}G(z^{-1})r(t), && (3) \\ y_e(t) &= \frac{B(z^{-1})}{A(z^{-1})}H(z^{-1})e(t) + \frac{1}{A(z^{-1})}V(z^{-1})e(t), \end{aligned}$$

$$\Big(A(z^{-1}) = 1 + a_1^o z^{-1} + \ldots + a_{n_a}^o z^{-n_a}, \quad B(z^{-1}) = b_1^o z^{-1} + \ldots + b_{n_b}^o z^{-n_b}\Big).$$

The predictor model used in identification is obtained from (1) by removing $v(t)$ and replacing ϑ^o with a generic parameter ϑ:

$$\widehat{y}(t,\vartheta) = \varphi(t)'\vartheta, \quad \vartheta \in \mathbb{R}^n. \tag{4}$$

The estimate $\widehat{\vartheta}_N$ is computed as:

$$\widehat{\vartheta}_N = \mathrm{sol}\left\{\frac{1}{N}\sum_{t=1}^{N}\zeta(t)\varphi(t)'\vartheta = \frac{1}{N}\sum_{t=1}^{N}\zeta(t)y(t)\right\}, \tag{5}$$

where N is the number of data points and $\zeta(t)$, the so called *instrumental variable*, is a n-dimensional, stationary, stochastic process, independent of $e(t)$.

Remark 1: Assuming that the data are generated according to (1) implies in a certain sense, precisely addressed later, that the true system belongs to the model class (4). This assumption is common whenever the asymptotic theory is developed for model quality assessment since, otherwise, the asymptotic theory can ascertain the variance part of the system-model mismatch only. See [9] and [12]. $\square$

Throughout the paper we assume that $\zeta(t)$ is chosen as follows:

Assumption 2: $\zeta(t) = \varphi_r(t)$, where $\varphi_r(t)$ is equal to

$$[y_r(t-1)\ \ldots\ y_r(t-n_a) \quad u_r(t-1)\ \ldots\ u_r(t-n_b)]',$$

i.e. it is the part of the observation vector depending on the external input sequence $r(t)$ as defined in (2) and (3). $\square$

Remark 2: The choice $\zeta(t) = \varphi_r(t)$ is optimal in a sense, i.e. it minimizes the estimation error variance (see [11]). In practical applications $\varphi_r(t)$ can be constructed approximately by first identifying an initial model (through some identification method) and then by generating "synthetic" data by feeding the identified model with $r(t)$ (open-loop case) or by feeding the entire control scheme where the plant has been substituted by the estimated model (closed-loop case). This procedure can be also repeated in an iterative scheme. $\square$

Let Θ^* be the set of solutions to equation

$$\mathbb{E}[\zeta(t)\varphi(t)']\vartheta = \mathbb{E}[\zeta(t)y(t)]. \tag{6}$$

It can be proved (see [9], [11] and [12]) that, in the present setting, the distance between $\widehat{\vartheta}_N$ and Θ^* tends to zero, as $N \to \infty$.

Moreover, thanks to Assumption 2 and equation (1), equation (6) can be rewritten as

$$\mathbb{E}[\varphi_r(t)\varphi(t)']\vartheta = \mathbb{E}[\varphi_r(t)\varphi(t)']\vartheta^o + \mathbb{E}[\varphi_r(t)v(t)],$$

and, since $\varphi(t) = \varphi_r(t) + \varphi_e(t)$ and $r(t)$ is independent of $e(t)$, the last equation is equivalent to

$$\mathbb{E}[\varphi_r(t)\varphi_r(t)'](\vartheta - \vartheta^o) = 0. \tag{7}$$

It follows that the cardinality of Θ^* depends on the rank of the matrix $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ and ϑ^o always belongs to Θ^*. Thus, if $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ is nonsingular, then Θ^* is the singleton $\{\vartheta^o\}$ and $\widehat{\vartheta}_N \to \vartheta^o$ as $N \to \infty$.

B. Asymptotic theory

We turn now to the problem of evaluating the accuracy of the model estimated through the *IV* method. The asymptotic Theorem 1 below can be trivially obtained from the general results presented in [9], [11] and [12]. Before the theorem we need some preliminaries.

Suppose that $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ is nonsingular. Then, let

$$Q_\alpha = \lambda^2\mathbb{E}\Big[\varphi_r^\alpha(t)\varphi_r^\alpha(t)'\Big], \tag{8}$$

where $\varphi_r^\alpha(t) = \sum_{i=0}^{\infty}\alpha_i\varphi_r(t-i)$ and α_i are the Markov coefficients of $V(z^{-1})$, viz. $V(z^{-1}) = \sum_{i=0}^{\infty}\alpha_i z^{-i}$, and let

$$P_\alpha = \mathbb{E}\Big[\varphi_r(t)\varphi_r(t)'\Big]^{-1}\cdot Q_\alpha\cdot\mathbb{E}\Big[\varphi_r(t)\varphi_r(t)'\Big]^{-1}. \tag{9}$$

Further, consider the following ellipsoid centered in $\widehat{\vartheta}_N$:

$$\mathcal{E}_\alpha(r) = \Big\{\vartheta : (\widehat{\vartheta}_N - \vartheta)'P_\alpha^{-1}(\widehat{\vartheta}_N - \vartheta) \le r\Big\}, \tag{10}$$

where r, the size of the ellipsoid, is a real positive number.

Remark 3: It is perhaps worth mentioning that assuming that $V(z^{-1})$ has no zeroes on the unit circle (Assumption 1) serves to the purpose to guarantee that the definition of $\mathcal{E}_\alpha(r)$ is well posed, i.e. P_α is invertible. See "Complements to Remark 3" in the appendix for details. $\square$

The following theorem suggests how to select r so that $\mathcal{E}_\alpha(r)$ is an ellipsoidal confidence region for ϑ^o of pre-assigned asymptotic probability ($\mathbb{P}\{E\}$ = probability of E in the following).

Theorem 1: Under the assumptions in this section, we have that

$$\lim_{N\to\infty} \mathbb{P}\left\{\vartheta^o \in \mathcal{E}_\alpha\left(\frac{\rho(p)}{N}\right)\right\} = p,$$

where $\rho(p)$ is the inverse of the function $p = \int_0^\rho f_{\chi^2}(x)\mathrm{d}x$ and $f_{\chi^2}(x)$ is the probability density of a χ^2 random variable with n degrees of freedom. $\square$

Remark 4: In the practical computation of $\mathcal{E}_\alpha(r)$, as it is obvious, Q_α and P_α cannot be computed exactly and have to be substituted by their sample counterparts, $\widehat{Q}_\alpha$ and $\widehat{P}_\alpha$. These can be computed simply by substituting $\mathbb{E}$ with $\frac{1}{N}\sum$, in equations (8) and (9). Note that, since $\widehat{Q}_\alpha$ and $\widehat{P}_\alpha$ tends almost surely to Q_α and P_α as N increases, the introduced approximation is negligible, provide that N is sufficiently large. $\square$

C. Discussion on the practical use of the asymptotic theory

It should be noted that the exact computation of $\mathcal{E}_\alpha\left(\frac{\rho(p)}{N}\right)$ requires the knowledge of λ^2 and $V(z^{-1})$ (see equations (8)–(10)). However, both these quantities are unknown in practice and have to be identified from the data.

To estimate λ^2 and $V(z^{-1})$, a common choice is to identify an ARMA model describing the residual error $\varepsilon(t,\widehat{\vartheta}_N) = y(t)-\widehat{y}(t,\widehat{\vartheta}_N)$. This is motivated by the fact that $\varepsilon(t,\widehat{\vartheta}_N) \to v(t) = V(z^{-1})e(t)$ as $N\to\infty$, since, under the assumption of Theorem 1, $\widehat{\vartheta}_N \to \vartheta^o$.

In a practical application, the number of data points is finite so that $\widehat{\vartheta}_N \neq \vartheta^o$ and λ^2 and $V(z^{-1})$ cannot be identified exactly. However, when $\varphi_r(t)$ is well exciting (and therefore $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ is positive definite with all the eigenvalues away from zero) we have $\widehat{\vartheta}_N \approx \vartheta^o$ and the introduced approximation is small.

Consider now the situation of poorly exciting inputs, so that matrix $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ has some eigenvalues close to zero. As long as $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ is not exactly singular, it is still true that the estimate $\widehat{\vartheta}_N$ converges to the true system parameter ϑ^o as $N\to\infty$. However, such a convergence takes place with a very slow rate and it may happen that $\widehat{\vartheta}_N$ is far from ϑ^o even for a large N. In this case it is no longer true that $\varepsilon(t,\widehat{\vartheta}_N)$ approximates $v(t)$, so that λ^2 and $V(z^{-1})$ cannot be identified with a good precision as well.

Thus, one could doubt as to the sensibility of applying the asymptotic result in case of poor excitation. One of the main scopes of this paper is to present the somehow surprisingly result that this is not so.

To this aim, we first develop in the next section a new asymptotic theory valid for the singular case (lack of excitation) and, then, we show in Section IV that, in the light of this new theory, the asymptotic results maintain its applicability in case of poor excitation.

III. Asymptotic theory for the singular case

Let us assume now that $\det \mathbb{E}[\varphi_r(t)\varphi_r(t)'] = 0$, i.e. we are in the singular case. The aim of this section is to show that a result similar to Theorem 1 still holds true in the present situation.

As it has been already noted in Section II, if matrix $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ is singular, then the set of asymptotic estimates Θ^* is not a singleton, but it is an affine subspace whose dimensionality d is equal to the dimension of the kernel of $\mathbb{E}[\varphi_r(t)\varphi_r(t)']$ (see equation (7)). Refer the parameter space to a basis having the first d components parallel to Θ^*, and the remaining $n-d$ components orthogonal to Θ^*. Let x [z] be the first d [the remaining $n-d$] coordinates in this basis (see Figure 1 for a graphical representation in a bi-dimensional space). Thus,

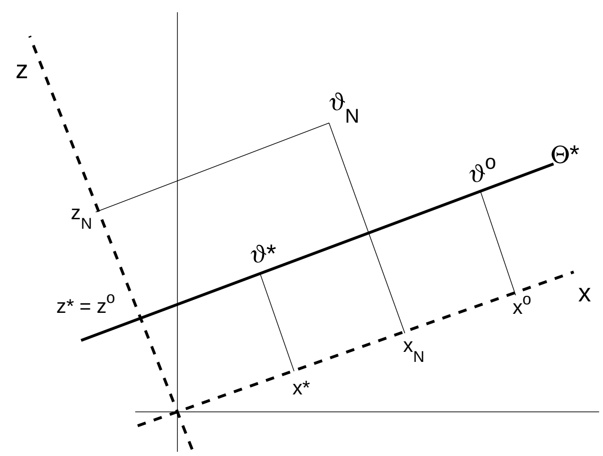

Fig. 1. The parameter space

$[(x^o)' \quad (z^o)']'$ and $[(\widehat{x}_N)' \quad (\widehat{z}_N)']'$ represent ϑ^o and $\widehat{\vartheta}_N$, respectively, and Θ^* writes $\{[x' \;\; z']' : z = z^o\}$.

In the present singular setting, matrix $\frac{1}{N}\sum\zeta(t)\varphi(t)' = \frac{1}{N}\sum\varphi_r(t)\varphi(t)'$ in equation (5) is singular itself, leaving a degree of freedom in the choice of $\widehat{\vartheta}_N$. In the sequel we assume that $\widehat{\vartheta}_N$ is fixed by a suitable deterministic tie-break rule (e.g select among the $\widehat{\vartheta}_N$ satisfying (5) the one which minimizes $\|\widehat{\vartheta}_N\|$) such that $\widehat{\vartheta}_N$ tends to a limiting estimate $\vartheta^* = [(x^*)' \;\; (z^*)']'$, as $N\to\infty$. Note that, though $\vartheta^*\in\Theta^*$ (and, therefore, $z^* = z^o$), $\vartheta^*\neq\vartheta^o$ in general since $x^*\neq x^o$.

We turn now our attention to the problem of model quality assessment. Since $\widehat{x}_N$ has been chosen deterministically, we cannot characterize the distance between $\widehat{x}_N$ and

x^o in a probabilistic way. In contrast, a probabilistic characterization is possible in the z direction, as precisely stated in Theorem 2 below.
We need a simple preliminary lemma. Let $[\varphi_r^x(t)' \;\; \varphi_r^z(t)']'$ be the vector $\varphi_r(t)$ referred to the x, z coordinates. We have:

Lemma 1: $\varphi_r^x(t) = 0$ almost surely, while $\mathbb{E}[\varphi_r^z(t)\varphi_r^z(t)']$ is nonsingular. Moreover, $\varepsilon(t, \vartheta^*) = y(t) - \widehat{y}(t, \vartheta^*)$ only depends on $e(t)$ and can be written as $\sum_{i=0}^{\infty} \beta_i e(t-i)$, for suitable β_i's. □

Proof: see the appendix.□

Let

$$Q_\beta^z = \lambda^2 \mathbb{E}\Big[\sum_{i=0}^{\infty} \beta_i \varphi_r^z(t-i) \sum_{j=0}^{\infty} \beta_j \varphi_r^z(t-j)'\Big],$$

and let

$$P_\beta^z = \mathbb{E}\Big[\varphi_r^z(t)\varphi_r^z(t)'\Big]^{-1} Q_\beta^z \mathbb{E}\Big[\varphi_r^z(t)\varphi_r^z(t)'\Big]^{-1},$$

and further consider the following ellipsoid centered in $\widehat{z}_N$:

$$\mathcal{E}_\beta^z(r) = \Big\{ z : (\widehat{z}_N - z)'(P_\beta^z)^{-1}(\widehat{z}_N - z) \le r \Big\},$$

where r is again the size of the ellipsoid and is a real positive number.

Remark 5: In view of Lemma 1 $\mathbb{E}\Big[\varphi_r^z(t)\varphi_r^z(t)'\Big]$ is invertible. Instead, similarly to Q_α in Remark 3, invertibility of Q_β^z requires that $\sum_{i=0}^{\infty} \beta_i z^{-i}$ has no zeroes on the unit circle. Such condition is assumed here for granted. □
The following theorem suggests how to select r so that $\mathcal{E}_\beta^z(r)$ is an ellipsoidal confidence region for z^o of pre-assigned asymptotic probability.

Theorem 2: We have that

$$\lim_{N \to \infty} \mathbb{P}\Big\{ z^o \in \mathcal{E}_\beta^z\Big(\frac{\rho(p)}{N}\Big) \Big\} = p,$$

where $\rho(p)$ is the inverse of the function $p = \int_0^\rho f_{\chi^2}(x)\mathrm{d}x$ and $f_{\chi^2}(x)$ is the probability density of a χ^2 random variable with $n-d$ degrees of freedom. □

Proof: see the appendix.□

Note that $\beta_i \neq \alpha_i$ in general. Thus, if one uses the Markov coefficients α_i's of $V(z^{-1})$ when computing $\mathcal{E}_\beta^z$, the resulting ellipsoid fails to represent a confidence region with the pre-assigned level of confidence. What is remarkable in Theorem 2 is that, in order to compute a correct ellipsoid, one has to use alternative coefficients β_i's and these coefficients can be in fact estimated from the residual error $\varepsilon(t, \widehat{\vartheta}_N)$ since $\widehat{\vartheta}_N \to \vartheta^*$ and $\varepsilon(t, \vartheta^*) = \sum_{i=0}^{\infty} \beta_i e(t-i)$ (see Lemma 1).

Remark 6: It is worth mentioning that Theorem 2 is a generalization of Theorem 1. As a matter of fact, in the nonsingular case, $d = 0$ so that $z = \vartheta$ and the statement of Theorem 2 reduces to that of Theorem 1. □

Remark 7: In view of the result of Theorem 2, it is possible to determine a confidence region for ϑ^o (and not only for z^o). Since the difference between $\widehat{x}_N$ and x^o remains unpredictable, the natural choice is to consider the degenerate ellipsoid

$$\mathcal{DE}_\beta\Big(\frac{\rho(p)}{N}\Big) = \Big\{ [x' \;\; z']' : (\widehat{z}_N - z)'(P_\beta^z)^{-1}(\widehat{z}_N - z) \le \frac{\rho(p)}{N} \Big\},$$

which is nothing but the ellipsoid $\mathcal{E}_\beta^z\Big(\frac{\rho(p)}{N}\Big)$ extended along the x direction towards infinity. The fact that $\mathcal{DE}_\beta\Big(\frac{\rho(p)}{N}\Big)$ is an asymptotic p-confidence region for ϑ^o directly follows from Theorem 2. □

IV. USE OF THE NEW ASYMPTOTIC RESULTS IN PRACTICE

Consider an identification problem where we have a finite number N of data points. After estimating $\widehat{\vartheta}_N$, we can compute the prediction error $\varepsilon(t, \widehat{\vartheta}_N)$ and then estimate a model $\sum_{i=1}^{\infty} \gamma_i e(t-i)$ describing such a prediction error. Here, γ_i's are the coefficients estimated from data and depending on the context in the discussion to follow, represent either an estimate of the α_i's or an estimate of the β_i's. Then, we compute the ellipsoid $\widehat{\mathcal{E}}_\gamma\Big(\frac{\rho(p)}{N}\Big)$ along the line traced in Section II. Namely,

$$\widehat{\mathcal{E}}_\gamma\Big(\frac{\rho(p)}{N}\Big) = \Big\{ \vartheta : (\widehat{\vartheta}_N - \vartheta)'\widehat{P}_\gamma^{-1}(\widehat{\vartheta}_N - \vartheta) \le \frac{\rho(p)}{N} \Big\}, \quad (11)$$

where

$$\begin{aligned} \widehat{P}_\gamma \;\; = \;\; & \Big(\tfrac{1}{N}\textstyle\sum_{t=1}^{N} \varphi_r(t)\varphi_r(t)'\Big)^{-1} \cdot \tfrac{\lambda^2}{N} \textstyle\sum_{t=1}^{N} \varphi_r^\gamma(t)\varphi_r^\gamma(t)' \cdot \\ & \cdot \Big(\tfrac{1}{N}\textstyle\sum_{t=1}^{N} \varphi_r(t)\varphi_r(t)'\Big)^{-1}, \end{aligned}$$

$\varphi_r^\gamma(t) = \sum_{i=0}^{\infty} \gamma_i \varphi_r(t-i)$ and $\rho(p)$ is the inverse of the function $p = \int_0^\rho f_{\chi^2}(x)\mathrm{d}x$ where $f_{\chi^2}(x)$ is the probability density of a χ^2 random variable with n degrees of freedom. Note that this is nothing but the normal line of proceeding in the application of the asymptotic theory to Instrumental Variable techniques.
Suppose first that the regressor $\varphi_r(t)$ excites well all the directions in the parameters space (full excitation case). Then, $\widehat{\vartheta}_N \approx \vartheta^o$ so that the γ_i's become an estimate of the α_i's and since $\widehat{\mathcal{E}}_\gamma\Big(\frac{\rho(p)}{N}\Big) \approx \mathcal{E}_\alpha\Big(\frac{\rho(p)}{N}\Big)$ Theorem 1 applies to conclude that we have computed a reliable estimate of a p-confidence region for ϑ^o.
The crucial fact is that the way of proceeding in (11) is also motivated in case of poor excitation where $\widehat{\vartheta}_N$ is far from ϑ^o (the case where estimating the $\widehat{\vartheta}_N$-ϑ^o mismatch is in fact more significant) as we next discuss grounding our analysis on the theory developed in Section III.
The poor excitation case can be seen as a "perturbation" case with respect to the singular setting of Section III, so that $\widehat{\vartheta}_N$

can be seen as a perturbed version of ϑ^* in that section. As we have seen in Section III, $\varepsilon(t,\vartheta^*) = \sum_{i=0}^{\infty} \beta_i e(t-i)$ and the β_i's are in fact the coefficients to be used in the construction of $\mathcal{E}_\beta^z\left(\frac{\rho(p)}{N}\right)$ in Theorem 2. This motivates the use of the γ_i's (which are estimates of the β_i's) in the construction of $\widehat{\mathcal{E}}_\gamma\left(\frac{\rho(p)}{N}\right)$ in equation (11). Note also that $\rho(p)$ in (11) refers to a χ^2 with n degrees of freedom (while we had $n-d$ degrees of freedom in Theorem 2) because in (11) we compute confidence regions for the whole n dimensional parameter vector ϑ^o.

V. Simulation results

The simulation example of the present section serves the purpose to illustrate the theory and it is not intended as a real application example. Correspondingly, the simplest possible situation has been selected. While the situation is artificial, the drawn conclusions bear a breath of general applicability. We have considered a first order data-generating system with $\vartheta^o = [-a^o \;\; b^o]' = [0.9 \;\; 0.1]'$ and $V(z^{-1}) = 1 + 0.5z^{-1}$. That is:

$$y(t) = 0.9y(t-1) + 0.1u(t-1) + e(t) + 0.5e(t-1),$$

where $e(t) = WGN(0,1)$ (WGN = White Gaussian Noise). To identify this system, the plant has been operated in open-loop with $u(t) = r(t)$, and the *IV* technique has been used with $\varphi(t) = [y(t-1) \;\; u(t-1)]'$ and $\zeta(t) = \varphi_r = [y_r(t-1) \;\; r(t-1)]'$, where

$$y_r(t) = 0.9y_r(t-1) + 0.1r(t-1).$$

As input signal, we have used $u(t) = 1 + \xi(t)$, where $\xi(t) = WGN(0, 10^{-6})$. Note that the variance of $\xi(t)$ is very small as compared to the noise variance so that the input $u(t)$ is poorly exciting ($u(t)$ is nearly exciting of order 1 while two parameters have to be identified).

The identification has been performed 500 times, by using $N = 5000$ data points each time. In each experiment a parameter vector $\widehat{\vartheta}_N^k = [-\widehat{a}_N^k \;\; \widehat{b}_N^k]'$, $k = 1\ldots500$, has been identified and a 95%-confidence region has been estimated as $\widehat{\mathcal{E}}_\gamma^k\left(\frac{\rho(0.95)}{N}\right)$ (the coefficients γ_i's have been computed by identifying an ARMA(3,3) model for the residue – see Section IV). The true parameter ϑ^o turned out to belong to $\widehat{\mathcal{E}}_\gamma^k\left(\frac{\rho(0.95)}{N}\right)$ in 491 cases out of 500, that is, with empirical frequency of 98%.

As an interesting comparison, we have further computed the 95% confidence region with the true parameters α_i's ($\sum_{i=0}^{\infty} \alpha_i z^{-i} = V(z^{-1})$). In this case, the success rate of $\vartheta^o \in \widehat{\mathcal{E}}_\alpha^k\left(\frac{\rho(0.95)}{N}\right)$ was of 47%. The results are summarized in Table I. As it appears, using the true parameters α_i's leads to wrong results.

VI. Concluding remarks

In this paper, a new asymptotic result, valid also in a singular case, has been developed for an *IV* identification setting. Grounded on this new result, we have shown that the asymptotic theory can be safely used for model quality assessment, even in the case of poor excitation and moderate data samples.

	ϑ^o in $\mathcal{E}$	ϑ^o out of $\mathcal{E}$	% of success
$\widehat{\mathcal{E}}_\alpha\left(\frac{\rho(0.95)}{N}\right)$	233	267	47%
$\widehat{\mathcal{E}}_\gamma\left(\frac{\rho(0.95)}{N}\right)$	491	9	98%

TABLE I

RELIABILITY OF THE ESTIMATED CONFIDENCE REGION

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VIII. Appendix

Complements to Remark 3: Note that, P_α is invertible provided that Q_α is nonsingular. Here, we show that $Q_\alpha > 0$.
Let v be a generic vector of $\mathbb{R}^n$ and consider $v'Q_\alpha v = \lambda^2 \cdot v'\mathbb{E}[\varphi_r^\alpha(t)\varphi_r^\alpha(t)']v = \lambda^2 \cdot v'\mathbb{E}[(\varphi_r^\alpha(t) - \mathbb{E}[\varphi_r^\alpha(t)])(\varphi_r^\alpha(t) - \mathbb{E}[\varphi_r^\alpha(t)])']v + \lambda^2 \cdot v'\mathbb{E}[\varphi_r^\alpha(t)]\mathbb{E}[\varphi_r^\alpha(t)]'v$.
Since $\varphi_r^\alpha(t) = V(z^{-1})\varphi_r(t)$, we obtain, through the Parseval identity,

$$\begin{aligned} v'Q_\alpha v &= \tfrac{\lambda^2}{2\pi}\int_{-\pi}^{\pi} v'\Phi_r(e^{j\omega})v \cdot |V(e^{j\omega})|^2 \mathrm{d}\omega \\ &\quad + \lambda^2 V(1)^2 \cdot v'\mathbb{E}[\varphi_r(t)]\mathbb{E}[\varphi_r(t)]'v, \end{aligned}$$

where $\Phi_r(e^{j\omega})$ is the spectrum of the n-dimensional process $\varphi_r(t)$.
This implies that

$$\begin{aligned} v'Q_\alpha v &\geq \min_{\omega\in[-\pi,\pi]} \Big\{ |V(e^{j\omega})|^2 \cdot \tfrac{\lambda^2}{2\pi}\int_{-\pi}^{\pi} v'\Phi_r(e^{j\omega})v \, \mathrm{d}\omega \\ &\quad + \lambda^2 V(1)^2 \cdot v'\mathbb{E}[\varphi_r(t)]\mathbb{E}[\varphi_r(t)]'v \Big\}. \end{aligned}$$

Applying now the assumption that $V(z^{-1})$ has no zeroes on the unit circle we have $\min_{\omega\in[-\pi,\pi]} |V(e^{j\omega})|^2 = k > 0$. Since, in addition, $\lambda^2 > 0$ and $E[\varphi_r(t)\varphi_r(t)'] > 0$ by assumption, we conclude that

$$\begin{aligned} v'Q_\alpha v &\geq k\lambda^2 \Big(\tfrac{1}{2\pi}\int_{-\pi}^{\pi} v'\Phi_r(e^{j\omega})v \, \mathrm{d}\omega \\ &\quad + v'\mathbb{E}[\varphi_r(t)]\mathbb{E}[\varphi_r(t)]'v \Big) \\ &= k\lambda^2 \cdot v'\mathbb{E}[\varphi_r(t)\varphi_r(t)']v > 0, \quad \forall v \neq 0, \end{aligned}$$

i.e. Q_α is positive definite. $\square$

Proof of Lemma 1: Let T the $n \times n$ rotation matrix such that $T\vartheta = [x' \;\; z']'$. Referring equation (7) to the x, z coordinates (i.e. $T\mathbb{E}[\varphi_r(t)\varphi_r(t)']T'T(\vartheta - \vartheta^o) = 0$), we obtain

$$\mathbb{E}\begin{bmatrix} \varphi_r^x(t)\varphi_r^x(t)' & \varphi_r^x(t)\varphi_r^z(t)' \\ \varphi_r^z(t)\varphi_r^x(t)' & \varphi_r^z(t)\varphi_r^z(t)' \end{bmatrix} \begin{bmatrix} x - x^o \\ z - z^o \end{bmatrix} = 0.$$

Since $[x' \;\; z']'$ is a solution of this equation if and only if $z = z^o$, while each value of x is feasible, it follows that $\mathbb{E}[\varphi_r^z(t)\varphi_r^z(t)']$ must be nonsingular, while $\mathbb{E}[\varphi_r^x(t)\varphi_r^x(t)']$ must be equal to zero so that $\varphi_r^x(t) = 0$, almost surely.
Consider now $\varepsilon(t, \vartheta^*) = y(t) - \widehat{y}(t, \vartheta^*)$. It can be rewritten as

$$\varphi(t)'(\vartheta^o - \vartheta^*) + v(t) = \varphi^x(t)'(x^o - x^*) + \varphi^z(t)'(z^o - z^*) + v(t),$$

where $[\varphi^x(t)' \;\; \varphi^z(t)']' = T\varphi(t)$. Noting that $z^o = z^*$ and that $\varphi^x(t)' = \varphi_r^x(t)' + \varphi_e^x(t)' = \varphi_e^x(t)'$ almost surely, we obtain

$$\varepsilon(t, \vartheta^*) = \varphi_e^x(t)'(x^o - x^*) + v(t). \tag{12}$$

Thus, $\varepsilon(t, \vartheta^*)$ is the stationary output of a dynamical linear system fed by $e(t)$, and $\sum_{i=0}^{\infty} \beta_i e(t - i)$ is the Markov representation of such a process. $\square$

Proof of Theorem 2: Referring equation (5) to the x, z coordinates, we have that

$$\tfrac{1}{N}\textstyle\sum_{t=1}^{N} \varphi_r^x(t)\varphi^x(t)'(\widehat{x}_N - x^o) + \tfrac{1}{N}\sum_{t=1}^{N} \varphi_r^x(t)\varphi^z(t)' \cdot \\ \cdot(\widehat{z}_N - z^o) = \tfrac{1}{N}\sum_{t=1}^{N} \varphi_r^x(t)v(t)$$

$$\tfrac{1}{N}\textstyle\sum_{t=1}^{N} \varphi_r^z(t)\varphi^x(t)'(\widehat{x}_N - x^o) + \tfrac{1}{N}\sum_{t=1}^{N} \varphi_r^z(t)\varphi^z(t)' \cdot \\ \cdot(\widehat{z}_N - z^o) = \tfrac{1}{N}\sum_{t=1}^{N} \varphi_r^z(t)v(t)$$

with $\varphi^x(t)$ and $\varphi^z(t)$ defined as in the proof of Lemma 1.
The first equation is $0 = 0$ almost surely, since $\varphi_r^x(t) = 0$, almost surely. Instead, inflating the second equation by $\sqrt{N}$ yields

$$\frac{1}{N}\sum_{t=1}^{N} \varphi_r^z(t)\varphi^z(t)'\sqrt{N}(\widehat{z}_N - z^o) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} \varphi_r^z(t)\widetilde{v}(t), \tag{13}$$

almost surely, where $\widetilde{v}(t) = v(t) + \varphi_e^x(t)'(x^o - \widehat{x}_N)$ and we have used the fact that $\varphi^x(t) = \varphi_r^x(t) + \varphi_e^x(t) = \varphi_e^x(t)$, almost surely. Note that the term $v + \varphi_e^x(t)'(x^o - \widehat{x}_N)$ depends just on $e(t)$ and, since $\widehat{x}_N \to x^*$ as $N \to \infty$, it tends to $\varepsilon(t, \vartheta^*)$ (see (12)). The latter, in turn, is equal to $\sum_{i=0}^{\infty} \beta_i e(t - i)$ (Lemma 1).
Then, following the same rationale in [9] – chapter 9 – it can be proved that the term $\sqrt{N}(\widehat{z}_N - z^o)$ in (13) is asymptotically distributed as a $(n - d)$-dimensional Gaussian random variable with zero mean and variance equal to

$$\mathbb{E}[\varphi_r^z(t)\varphi_r^z(t)']^{-1} \cdot \lambda^2 \mathbb{E}\Big[\textstyle\sum_{i=0}^{\infty} \beta_i \varphi_r^z(t - i) \cdot \\ \cdot \textstyle\sum_{j=0}^{\infty} \beta_j \varphi_r^z(t - j)'\Big] \cdot \cdot \mathbb{E}[\varphi_r^z(t)\varphi_r^z(t)']^{-1} = P_\beta^z.$$

Then, the theorem thesis easily follows noting that $N(\widehat{z}_N - z)'(P_\beta^z)^{-1}(\widehat{z}_N - z)$ is asymptotically distributed as a χ^2 random variable with $(n - d)$-degree of freedom. $\square$

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