Technical Notes and Correspondence

Adaptive RLS Algorithms Under Stochastic Excitation—L^2 Convergence Analysis

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Abstract—**In this note, the RLS algorithm with forgetting factor is considered. The basic assumptions are that the data generation mechanism is free of disturbances and that the observation vector is a stochastic process satisfying a ϕ-mixing condition. A stochastic characterization of persistent excitation is first given. Then, it is proven that the algorithm is exponentially convergent in mean-square sense.**

I. Introduction

In the area of identification via regression type models, the least mean-square (LMS) algorithm has been extensively analyzed and a consistent set of theoretical results is now available, see, e.g., [1]-[8]. On the contrary, the analysis of adaptive recursive least squares (RLS) algorithms is far from being satisfactorily developed.

As is well known, the classical RLS algorithm can be given an adaptive format by incorporating a forgetting factor (FF) device. The FF is a parameter belonging to the interval (0, 1] that can be used to modulate the algorithm memory length. Precisely, by decreasing the value of FF, the memory length can be shortened. On the contrary, by setting FF = 1, the forgetting factor gadget is excluded, so reducing to the classical RLS algorithm. Various types of time-varying laws for the FF have been proposed and studied. A family of RLS variants is characterized by an FF that asymptotically saturates to 1; consequently, the asymptotic behavior of algorithms of this class is close to that of the classical RLS. A truly adaptive RLS variant requires an FF which cannot saturate to 1. In the so-called exponential forgetting (EF) variant, this is achieved by simply letting FF = constant < 1. In other cases, the value of the FF is selected so as to tune the memory length according to the dynamics of the parameters. For instance, in [9] the value of the FF is determined on the basis of the prediction error, so that the corresponding algorithm will be referred to as PEF (prediction error forgetting). An interesting alternative is the one proposed in [10]-[13], where the discount effect is nonuniformly distributed in the parameter space, so leading to the directional forgetting (DF) algorithm.

The theoretical analysis of all these algorithms can be carried out by making reference to various types of data generation mechanisms. In the *deterministic context* it is assumed that the data generation mechanism is subject to no disturbances and that the input variables satisfy a persistent excitation condition stated in deterministic terms. In such a framework, a number of convergence results have been worked out, see [14] for EF, [15] for PEF, and [16], [7] for DF. As for the *stochastic context*, a number of results are available for the classical RLS algorithm. In particular, by making reference to a data generation mechanism subject to constant parameter and stochastic disturbances, the almost sure convergence of the classical RLS algorithm is proven in [18] via martingale theory, see also [19]. Roughly, in this case, the convergence is due to the accumulation of available information carried by data as time increases. A similar situation arises if one considers those RLS variants with an FF which saturates to 1. (For the analysis of these algorithms, see e.g., [20]-[22].)

In adaptive RLS, the discount of past data prevents the information accumulation. Therefore, if the data generation mechanism is subject to disturbances, one cannot expect that the parameter estimation error vanishes (see, e.g., [17] for a discussion relative to an EF type case). To the best knowledge of the present authors, the area of adaptive RLS is lacking in theoretical results and still requires a considerable investigation effort.

This note deals with the convergence of a fairly general class of adaptive RLS algorithms precisely defined by the following equations:

$$\epsilon(t) = y(t) - \varphi(t)'\hat{\vartheta}(t-1) \tag{1a}$$

$$r(t) = \varphi(t)'P(t-1)\varphi(t) \tag{1b}$$

$$K(t) = \frac{P(t-1)\varphi(t)}{1+r(t)} \tag{1c}$$

$$\hat{\vartheta}(t) = \hat{\vartheta}(t-1) + K(t)\epsilon(t) \tag{1d}$$

$$P(t) = \frac{1}{\mu(t)}\left[P(t-1) - \frac{P(t-1)\varphi(t)\varphi(t)'P(t-1)}{1+r(t)}\right] \tag{1e}$$

where the forgetting factor $\mu(t)$ is such that

$$0 < \mu_0 \le \mu(t) \le \mu_1 < 1. \tag{1f}$$

As usual, these equations make reference to the linear regression model, with $\varphi(t) \in \mathbb{R}^n$ as observation vector and $y(t) \in \mathbb{R}^1$ as observed output. Note that constraint (1f) on the forgetting factor is mild enough to encompass as particular cases the most popular forgetting factor algorithms proposed in the literature. In the subsequent analysis, $\mu(\cdot)$ can be either a deterministic sequence or a stochastic process. The latter point of view is to be taken whenever $\mu(t)$ is tuned according to the observed value of the prediction error. For easy reference, (1) will be referred to as the variable forgetting (VF) algorithm.

Our analysis will be developed assuming that the data are generated as follows:

$$y(t) = \varphi(t)'\vartheta^\circ$$

where ϑ° is deterministic and $\varphi(t)$ is a stochastic process. In engineering and physical situations, one would expect that the present dynamics of data is weakly dependent upon the remote past, and even that the present-past ties vanish asymptotically. Therefore it appears no limitation to impose some asymptotic independence condition on $\varphi(\cdot)$. A possibility is then to assume that $\varphi(\cdot)$ satisfies an ergodicity assumption, see [23], [3]. Indeed, ergodicity can be seen as a characterization of asymptotic independence. However, this assumption applies to stationary processes only. In this note, we

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will instead resort to the so-called ϕ-mixing conditions, which impose a form of asymptotic independence of events with increase time separation. Such a condition can be applied to nonstationary processes as well, and includes the special case when $\varphi(t)$ and $\varphi(\tau)$ are independent each other for $|t-\tau|$ large enough [2], [6].

The remainder of the note is organized in two sections, as follows: in Section II, a nice stochastic characterization of the notion of persistent excitation is derived, then in Section III, it is shown that the estimate $\hat{\vartheta}(t)$ tends exponentially to ϑ° in mean square.

The convergence analysis provided herein is believed to be a significant step forward in the stochastic analysis of the adaptive RLS algorithm. Indeed, the proposed approach looks promising to deal with more complex data generation mechanism, where the parameters are subject to a deterministic or stochastic drift and the output is corrupted by additive noise. These cases will be analyzed in forthcoming papers.

II. A Stochastic Notion of Persistent Excitation

A. The ϕ-Mixing Notion

The definition of ϕ-mixing stochastic process is based on the concept of dependence coefficient originally proposed in [24]. We will now concisely introduce the definitions of dependence coefficient and ϕ-mixing process.

With reference to a probability space $(\Omega, \mathfrak{F}, \mathfrak{p})$, consider two σ-algebras $\mathfrak{F}_1$ and $\mathfrak{F}_2$ contained in $\mathfrak{F}$. The dependence coefficient of $\mathfrak{F}_1$ with respect to $\mathfrak{F}_2$ is defined as

$$\phi(\mathfrak{F}_1, \mathfrak{F}_2) = \sup_{A \in \mathfrak{F}_2} \left\{ \operatorname*{ess\,sup}_{\omega \in \Omega} |\mathfrak{p}(A/\mathfrak{F}_1) - \mathfrak{p}(A)| \right\}.$$

Notice that $\phi(\mathfrak{F}_1, \mathfrak{F}_2) \in [0,1]$. In particular, $\phi(\mathfrak{F}_1, \mathfrak{F}_2) = 0$ when the σ-algebras are independent, while, assuming that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are "structured enough," $\phi(\mathfrak{F}_1, \mathfrak{F}_2)$ is close to unity when $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are "strongly dependent." Note that, in general, $\phi(\mathfrak{F}_1, \mathfrak{F}_2) \neq \phi(\mathfrak{F}_2, \mathfrak{F}_1)$. Moreover, it can be seen that the dependence coefficient satisfies the following *monotonicity property*: $\phi(\mathfrak{F}_1, \mathfrak{F}_2) \leq \phi(\mathfrak{E}_1, \mathfrak{E}_2)$ if $\mathfrak{E}_1 \supseteq \mathfrak{F}_1$ and $\mathfrak{E}_2 \supseteq \mathfrak{F}_2$, where $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are contained in $\mathfrak{F}$. Indeed, ess sup $|\mathfrak{p}(A/\mathfrak{F}_1) - \mathfrak{p}(A)|$ = ess sup $|E[I_A/\mathfrak{F}_1] - E[I_A/\mathfrak{F}_0]|$ where I_A is the indicator of A and $\mathfrak{F}_0 = \{\phi, \Omega\}$ is the trivial σ-algebra. Roughly, $E[I_A/\mathfrak{F}_i]$, $i = 0, 1$, is the random variable I_A "as seen" from the viewpoint of σ-algebra $\mathfrak{F}_i$. Therefore, esssup $|\mathfrak{p}(A/\mathfrak{F}_1) - \mathfrak{p}(A)|$ characterizes the difference between the trivial viewpoint ($\mathfrak{F}_0$) and the actual one ($\mathfrak{F}_1$). Consequently, it is expected that such a difference becomes larger and larger in the extent to which the σ-algebra $\mathfrak{F}_1$ becomes finer and finer. The interested reader is referred to [25] for an extensive presentation of the properties of the dependence coefficient, see also the Appendix for a few results of interest for this note.

Consider now a stochastic process $\xi(t)$, $t = 1, 2, \cdots$, defined over $(\Omega, \mathfrak{F}, \mathfrak{p})$. The dependence coefficient can be used to introduce a notion of asymptotic independence. Precisely, denoting by $\mathfrak{F}_r^s$ the σ-algebra generated by $\xi(r), \xi(r+1), \cdots, \xi(s)$, define $\rho_m = \sup\{\phi(\mathfrak{F}_1^i, \mathfrak{F}_{i+m}^{\infty})\}$ as the process dependence coefficient at lag m. Note that ρ_m can be roughly interpreted as a correlation coefficient between the past and the m-steps forward future. Among other things, the monotonicity property of the dependence coefficient implies that ρ_m is monotonically decreasing with m.

The notion of ϕ-mixing sequence can now be given.

Definition 1: The quantity $\delta = \sum_{1}^{\infty}{}_m \rho_m^{1/2}$ is said to be the process *dependence index*. The process is said to be *ϕ-mixing* if the associated dependence index is finite.

Remark: Let $\xi(t): \Omega \to \mathbb{R}^n$, $t = 1, 2, \cdots$, be a ϕ-mixing process with dependence index δ_ξ. An important consequence of the monotonicity property of the dependence coefficient is that, given any measurable transformation $g_t: \mathbb{R}^n \to \mathbb{R}^m$, $t = 1, 2, \cdots$, the process $\eta(t) = g_t(\xi(t))$ is ϕ-mixing too, with a dependence coefficient $\delta_\eta \leq \delta_\xi$. In other words, the ϕ-mixing property is automatically preserved under measurable transformations; a fact which plays a major role in the forthcoming analysis. □

Note that, due to its finite memory, a moving average process of order r is ϕ-mixing with $\delta \leq r$. In the autoregressive case, under the stability condition, the influence of the past values taken by the process on the value of the present random variable tails off. However, the σ-algebra generated by the process is monotonically increasing, so that the past and the forward future cannot be independent in terms of σ-algebras even asymptotically. Therefore, AR (or ARMA) processes do not generate ϕ-mixing observation vectors. The extension of the analysis provided below to cope with ARMA processes would require the definition of a new ϕ-mixing type notion, involving the values taken by the process besides its σ-algebras. This extension is currently underway. On the other hand, as observed in [23], the ϕ-mixing condition is less presumptuous than the acceptance of many other conditions currently used in the literature, and despite its limitations, it appears as a reasonable type of condition in engineering contexts.

B. Persistent Excitation

We are now in a position to provide a stochastic characterization of the persistent excitation notion. With reference to a sequence of n-dimensional random vectors $\varphi(t)$, $t = 1, 2, \cdots$, such a characterization calls for the existence of an integer s and a real $\alpha > 0$ such that

$$E\left[\lambda_{\min}\left\{\sum_{t+1^i}^{t+s} \varphi(i)\varphi(i)'\right\}\right] \geq \alpha, \qquad \forall t. \tag{2}$$

Theorem 1: Assume that

1) there exists $a > 0$ such that $E[\varphi(t)\varphi(t)'] \geq aI$, $\forall t$;
2) there exists b such that $E[(\varphi(t)'\varphi(t))^2] \leq b$, $\forall t$;
3) $\{\varphi(t)\}$ is ϕ-mixing.

Then, there exist an integer $s > 0$ and a real $\alpha > 0$ such that (2) holds.

Proof: For each 1, $k \in [1, n]$, let $\xi_{1,k}(t) = u_1(t)u_k(t) - E[u_1(t)u_k(t)]$, where $u_j(t)$, $j = 1, 2, \cdots, n$ is the jth component of $\varphi(t)$. Since $E[\varphi(t)'\varphi(t))^2] \geq E[(u_1(t)u_k(t))^2] \geq E[\xi_{1,k}(t)^2]$, from 2 it follows that

$$E\left[\xi_{1,k}(t)^2\right] \leq b.$$

In view of Remark 1, the ϕ-mixing assumption of $\{\varphi(t)\}$ implies that the scalar sequence $\{\xi_{1,k}(t)\}$ is ϕ-mixing, too.

In conclusion, the sequence of scalar random variables $\{\xi_{1,k}(t)\}$ satisfies the assumptions of Lemma 4 of the Appendix. Consequently

$$\lim_{r \to \infty} \frac{1}{r^2} E\left[\gamma_{1,k}^2(r,t)\right] = 0, \quad \text{uniformly w.r.t. } t \tag{3}$$

where $\gamma_{1,k}(r,t) = \sum_{t+1^i}^{t+r} \xi_{l,u}(\mathrm{i})$.

By means of (3) and assumption 1) it is now possible to prove inequality (2). Indeed, for any $\omega \in \Omega$

$$\lambda_{\min}\left[\sum_{t+1^i}^{t+r} \varphi(i)\varphi(i)'\right] \geq \lambda_{\min}\left[\sum_{t+1^i}^{t+r} E\left[\varphi(i)\varphi(i)'\right]\right] - \|\Gamma(r,t)\| \tag{4}$$

where

$$\Gamma(r,t) = \sum_{t+1}^{t+r} \left[\varphi(i)\varphi(i)' - E[\varphi(i)\varphi(i)'] \right].$$

As for $\Gamma(r, t)$, observe that

$$E[\|\Gamma(r,t)\|]^2 \le E\left[\sum_{1\ l,u}^{n} |\gamma_{1,k}(r,t)|\right]^2 \le E\left[\left(\sum_{1\ l,u}^{n} |\gamma_{1,k}(r,t)|\right)^2\right].$$

Computing the square and using the Schwarz inequality, one obtains

$$E[\|\Gamma(r,t)\|]^2 \le n^4 \max_{1,k} E[\gamma_{1,k}(r,t)^2].$$

Therefore, (3) implies that

$$\lim_{r\to\infty} \frac{1}{r^2} E[\|\Gamma(r,t)\|]^2 = 0, \quad \text{uniformly w.r.t. } t \tag{5}$$

The statement follows from (4), (5) and assumption 1).

Remark 2: Inequality (2) supplies the characterization of *persistent excitation* used in Section III. Since assumption 1) of Theorem 1 can be seen as an L^2*-excitation* condition, Theorem 1 roughly says that, under the hypothesis of finiteness of the 4th-order moments (condition 2)), L^2-excitation plus ϕ-mixing entail persistent excitation. □

As is well known, the notion of persistent excitation has a long history in system identification, going back to the early 1970's, or even earlier, see [27] and [28]. The stochastic characterization has been recently discussed in a number of papers, such as [1], [2], and [6]. In an LMS context, characterizations similar to (2) are proposed in [23, Theorem 7] by relying on ergodicity assumptions. The stochastic notion of persistent excitation supplied in [2] and [6] coincides with our characterization. However, the assumptions of [2] and [6] are much stronger than those of Theorem 1 above. For instance, in [2, Lemma 3], it is assumed that there exists and integer m such that $\{\varphi(j) \mid j \le i\}$ and $\{\varphi(j) \mid j \ge i + m\}$ are independent for each i. This obviously implies, but is not implied by, the ϕ-mixing assumption. Moreover, in [2], the further assumption is made that there exists the moments of $\varphi(t)$ up to the order $24m$. Such an order may be very high whenever the data are correlated over a long interval of time.

III. Convergence of the RLS Algorithm

Letting

$$\tilde{\vartheta}(t) = \hat{\vartheta}(t) - \vartheta^{\circ},$$

the recursive equation for the parameter estimation error $\tilde{\vartheta}(t)$ is

$$\tilde{\vartheta}(t) = F(t)\tilde{\vartheta}(t-1) \tag{6a}$$

where

$$F(t) = I - \frac{P(t-1)\varphi(t)\varphi(t)'}{1 + r(t)}. \tag{6b}$$

Obviously, the convergence of $\tilde{\vartheta}(t)$ to zero depends upon the characteristics of the external variables entering the estimation algorithm, i.e., $\varphi(\cdot)$ and $\mu(\cdot)$.

The classes of external variables that will be used in the convergence theorem are defined as follows.

Definition 2: Given three real numbers a, b, and d, with $a > 0$, the set of pairs $(\varphi(\cdot), \mu(\cdot))$ such that

1) $E[\varphi(t)\varphi(t)'] \ge aI, \quad \forall t;$

2) $E[(\varphi(t)'\varphi(t))^2] \le b, \quad \forall t;$

3) $\mathrm{Col}[\varphi(\cdot), \mu(\cdot)]$ is *ϕ-mixing* with dependence index $(\delta < d$ is denoted as $\mathfrak{M}(a, b, d)$. □

In view of the proofs of Theorem 1 and Lemma 4, see [26], it can be seen that, for a class $\mathfrak{M}(a, b, d)$, there exists a pair (s, α) satisfying (2) for each $\varphi(\cdot)$ of the class. The minimum value of integer s for which there exists $\alpha > 0$ such that (2) holds for each element belonging to $\mathfrak{M}(a, b, d)$ is said to be the *order of persistent excitation* of the class.

The main technical tool in the derivation of the convergence result is the Lyapunov-type function

$$V(t) = \tilde{\vartheta}(t)'P(t)^{-1}\tilde{\vartheta}(t). \tag{7}$$

Note that, if one assumes that $P(0) > 0$, then (1e) and (1f) imply that $P(t) > 0$, $\forall t$. Moreover, from (1e)

$$P(t)^{-1} = \mu(t)P(t-1)^{-1} + \mu(t)\varphi(t)\varphi(t)'. \tag{8}$$

From (6) and (8), it is easy to see that

$$V(t) \le \mu(t)V(t-1)$$

so that $V(\cdot)$ is a monotonically decreasing sequence. This inequality, together with (7), entails that

$$\|\tilde{\vartheta}(t)\|^2 \le \frac{\prod_{i=1}^{t}\mu(i)}{\lambda_{\min}[P(t)^{-1}]} V(0). \tag{9}$$

The convergence issue of $\tilde{\vartheta}(t)$ will be addressed by analyzing the R.H.S. of (9).

As already observed, $\varphi(\cdot)$ and $\mu(\cdot)$ play the role of input variables for algorithm (1). Therefore, the convergence notion has to be defined w.r.t. a class $\mathfrak{C} = \{(\varphi(\cdot), \mu(\cdot))\}$.

Definition 3: Algorithm (1) is said to be exponentially convergent in L^2 w.r.t. the class $\mathfrak{C}$ if there exists $\beta > 0$ and $\nu \in (0, 1)$ such that, for any $(\varphi(\cdot), \mu(\cdot)) \in \mathfrak{C}$

$$E[\|\tilde{\vartheta}(t)\|^2] \le \beta\nu^t\|\tilde{\vartheta}(0)\|^2, \quad \forall\tilde{\vartheta}(0). \tag{10}$$ □

Note that it is required that β and ν do not depend on the particular pair $(\varphi(\cdot), \mu(\cdot))$ in $\mathfrak{C}$.

Theorem 2: Algorithm (1) is exponentially convergent in L^2 w.r.t. any class $\mathfrak{M}(a, b, d)$.

Proof: Denote by s the order of persistent excitation of the $\mathfrak{M}(a, b, d)$ class, and consider a real $\alpha > 0$ such that (2) holds. Observe that, for any sequence $\varphi(\cdot)$

$$\lambda_{\min}\left\{\sum_{t+1}^{t+s}\varphi(i)\varphi(i)'\right\} \le \sum_{t+1}^{t+s}\varphi(i)'\varphi(i).$$

Consequently, thanks to the Schwarz inequality and 2) of Definition 2

$$E\left[\left(\lambda_{\min}\left\{\sum_{t+1}^{t+s}\varphi(i)\varphi(i)'\right\}\right)^2\right] \le s^2 b, \quad \forall t. \tag{11}$$

Consider now the quantity $g(\alpha, \beta, z)$ where $g(\cdot,\cdot,\cdot)$ is the function defined in Lemma 5 of the Appendix, α has been defined above, $\beta = s^2 b$ and $z \in (0, 1)$. Since the sequence of process dependence coefficients ρ_m is monotonically decreasing and $\sum_{1}^{\infty} m\rho_m^{1/2} < d$ for each $\varphi(\cdot)$ in the $\mathfrak{M}(a, b, d)$ class, it is obvious that $\rho_m^{1/2} < d/m$ *for each* $\varphi(\cdot)$ of such a class. Therefore, since $g(\alpha, \beta, z) < 1$, there exists a positive integer h, independent of the particular sequence $\varphi(\cdot)$ in the considered class, such that

$$g(\alpha, \beta, z) + \rho_{hs} < 1. \tag{12}$$

Partition now the time axis in intervals of length $(h+1)s$ and define

$$T(m) = \{i \mid (m-1)(h+1)s + 1 \le i \le m(h+1)s\},$$
$$\Delta(m) = \{i \mid (m-1)(h+1)s + hs + 1 \le i \le m(h+1)s\}.$$

Observe that recursion (8) entails that

$$P(m(h+1)s)^{-1} \ge \bar{\mu}(m)\left[P((m-1)(h+1)s)^{-1} + \sum_{i\in\Delta(m)} \varphi(i)\varphi(i)'\right] \quad (13)$$

where

$$\bar{\mu}(m) = \prod_{i\in T(m)} \mu(i).$$

Let

$$M(m) = \begin{cases} 1, & m = 0 \\ \prod_{j=1}^{m} \bar{\mu}(j), & m \ge 1 \end{cases} \quad (14)$$

and

$$h(m) = \frac{M(m)}{\lambda_{\min}\left[P(m(h+1)s)^{-1}\right]}.$$

In view of (13), $\forall m \ge 1$

$$h(m) \le \frac{M(m-1)}{\lambda_{\min}\left[P((m-1)(h+1)s)^{-1}\right] + \lambda_{\min}\left[\sum_{i\in\Delta(m)} \varphi(i)\varphi(i)'\right]}.$$

Apply now Lemma 5 of the Appendix with $\xi = M(m-1)$, $\xi = \lambda_{\min}[P((m-1)(h+1)s)^{-1}]$, $\eta = \lambda_{\min}[\sum_{i\in\Delta(m)}\varphi(i)\varphi(i)']$, $\mathfrak{F}_1 = \mathfrak{F}_{(m-1)(h+1)s+hs+1}^{m(h+1)s}$, $\mathfrak{F}_2 = \mathfrak{F}_1^{(m-1)(h+1)s}$, where $\mathfrak{F}_j^k$ is the σ-algebra generated by $\{\varphi(i) \mid i \in [j,k]\}$. Obviously, η is $\mathfrak{F}_1$ measurable, has a positive expected value and, thanks to (11), satisfies condition b) of Lemma 5. Moreover $\xi \ge 0$ and $\zeta \ge 0$ are $\mathfrak{F}_2$ measurable. From (19)

$$E[h(m)] \le E[M(m-1)/z\alpha] + (g(\alpha,\beta,z) + \rho_{hs})E[h(m-1)] \quad (15)$$

where the relation $h(m-1) = \xi/\zeta$ has been used.

Recall that μ_1 is the upper bound for the forgetting factor (see (1f)) and let

$$\nu \overset{\text{def}}{=} \max\{g(\alpha,\beta,z) + \rho_{\text{hs}}, \mu_1^{(h+1)s}\}.$$

Inequality (15) can be given the form

$$E[h(m)] \le \frac{\nu^{m-1}}{z\alpha} + \nu E[h(m-1)]. \quad (16)$$

Solving (16), one obtains

$$E[h(m)] \le m\frac{\nu^{m-1}}{z\alpha} + \nu^m \frac{1}{\lambda_{\min}[P(0)^{-1}]}. \quad (17)$$

So far, let $t = m(h+1)s + \tau$, with $\tau \in [0, (h+1)s - 1]$. From (9) and definition (14) it follows that $\|\tilde{\vartheta}(t)\|^2 \le M(m)c(\tau)V(0)/\lambda_{\min}[P(t)^{-1}]$, where $c(\tau) = \prod_{i=m(h+1)s+1}^{m(h+1)s+\tau}\mu(i)$. On the other hand, (8) entails that $P(m(h+1)s+\tau)^{-1} \ge c(\tau)P(m(h+1)s)^{-1}$. Therefore

$$\|\tilde{\vartheta}(t)\|^2 \le h(m)V(0).$$

Therefore, from (17)

$$E[\|\tilde{\vartheta}(t)\|^2] \le \gamma^t(tc_1 + c_2)\|\tilde{\vartheta}(0)\|^2 \quad (18)$$

where

$$c_1 = (\nu^2 z\lambda_{\min}[P(0)](h+1)s\alpha)^{-1}, \quad c_2 = \frac{1}{\nu}\frac{\lambda_{\max}[P(0)]}{\lambda_{\min}[P(0)]},$$

$$\gamma = \nu^{1/(h+1)s}.$$

Since h has been selected so as to meet inequality (12), $\nu < 1$, so that $\gamma < 1$. Therefore, in the R.H.S. of (18), there is the product of a term which linearly increases with time by a term which converges exponentially to zero. Thus, by choosing any $\rho \in (\gamma, 1)$, there exists a constant β (obviously, depending on the value taken for ρ) such that (10) is satisfied.

IV. Concluding Remarks

A very general class of RLS algorithms with forgetting factor is considered in this note. An L^2-convergence result of the estimate to the true constant parametrization is proven under the assumption that the data generation mechanism is free of disturbances. The data is described as a (possibly nonstationary) stochastic process satisfying the classical ϕ-mixing condition. Although such a condition appears quite flexible, it excludes ARMA processes (with nontrivial AR part). However, it is likely that the convergence analysis provided herein can be generalized to include ARMA processes as well, by means of an extended ϕ-mixing notion. Furthermore, the approach presented in this note constitutes the basis for the study of data generation mechanisms with time-varying parameters and/or stochastic disturbances. This will be the subject of further research.

Appendix

A Miscellanea of Results on the ϕ-Mixing Notion

By means of the ϕ-mixing notion, a number of classical results of probability theory, derived under the independence assumption, can be given a more general statement. This is the case for the following lemmas; their proofs can be found in [25] (Lemmas 1 and 2) and in the report [26] for the remaining lemmas. Lemma 1 is used in the proof of Lemma 3, which, in turn, is useful to prove Lemma 5. As for Lemma 2, it is required in the proof of Lemma 4.

Lemmas:

1) Let A and B be two subsets of Ω belonging to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively. Then,

$$|\mathfrak{p}(A\cap B) - \mathfrak{p}(A)\mathfrak{p}(B)| \le \phi(\mathfrak{F}_1,\mathfrak{F}_2)\mathfrak{p}(A).$$

2) Let the random variables ξ_i be $\mathfrak{F}_i$ measurable, $i = 1,2$, and $E[|\xi_1|^p] < \infty$, $E[|\xi_2|^q] < \infty$, with $1/p + 1/q = 1$. Then

$$|E[\xi_1\xi_2] - E[\xi_1]E[\xi_2]| \le 2\phi^{1/p}(\mathfrak{F}_1,\mathfrak{F}_2)E^{1/p}[|\xi_1|^p]E^{1/q}[|\xi_2|^q].$$

3) Let $\xi \ge 0$ be a random variable measurable w.r.t. the σ-algebra $\mathfrak{F}_1 \subseteq \mathfrak{F}$ and denote by B an element of the σ-algebra $\mathfrak{F}_2 \subseteq \mathfrak{F}$. Then

$$\int_B \xi\, d\mathfrak{p} \le (\mathfrak{p}(B) + \phi(\mathfrak{F}_1,\mathfrak{F}_2))E[\xi].$$

4) Consider a sequence of scalar random variables $\{\xi(t)\}$ such that: a) $E[\xi(t)] = 0$, $\forall t$; b) There exists b such that $E[\xi(t)^2] \le b$, $\forall t$; c) $\{\xi(t)\}$ is ϕ-mixing. Then

$$\lim_{r\to\infty}\frac{1}{r^2}E\left[\left(\sum_{t+1}^{t+r}\xi(i)\right)^2\right] = 0, \quad \text{uniformly w.r.t. } t.$$

5) Consider a random variable $\eta \ge 0$ measurable w.r.t. the σ-algebra $\mathfrak{F}_1 \subseteq \mathfrak{F}$, such that: a) $E[\eta] \ge \alpha > 0$; b) $E[\eta^2] \le \beta < \infty$, and any real $z \in (0,1)$. If $\xi \ge 0$ and $\zeta > 0$ are two further random

variables, measurable w.r.t. $\mathfrak{F}_2 \subseteq \mathfrak{F}$, then

$$E\left[\frac{\xi}{\zeta+\eta}\right] \le E\left[\frac{\xi}{z\alpha}\right] + \left(g(\alpha, \beta, z) + \phi(\mathfrak{F}_2, \mathfrak{F}_1)\right) E\left[\frac{\xi}{\zeta}\right] \tag{19}$$

where $g(\alpha, \beta, z)$ is a suitable quantity such that $g(\alpha, \beta, z) < 1$.

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An Adaptive Impedance/Force Controller for Robot Manipulators

Ricardo Carelli and Rafael Kelly

Abstract—**An adaptive impedance/force controller for constrained robots with uncertain dynamic model parameters is presented. The controller has been designed based on singular model robot representation. Some properties of a reduced form robot model are exploited to show that global tracking is achieved in the sense that asymptotic approach to an impedance manifold and bounded force error are obtained.**

I. Introduction

Control or robotic manipulators can be categorized into two modes: motion control and constrained motion control. *Motion control* is used when the robot arm moves in a free space without interacting with the environment. *Constrained motion control* is concerned with the control of a robot whose end-effector mechanically interacts with the environment. Several approaches to constrained motion control have been suggested, such as impedance control, force control, and hybrid motion/force control.

The fundamental philosophy of *impedance control* [4], [6] is that the manipulator control system should be designed not only to track a motion or force trajectory, but rather to regulate the mechanical impedance of the manipulator. *Force control* may be required to complete a task. The force control problem consists of computing the applied joint force/torques for a manipulator so that its end effector can accurately apply a desired force/torque on its environment. The *hybrid position/force* control method was proposed by Raibert and Craig [12], where the position of the end effector must be controlled in certain coordinate directions of the work space and the force must be controlled in the remaining ones. In a paper by McClamroch and Wang [10], a theoretical framework is presented for the study of stability of robots under constrained tasks. Singular model robot representation is used with the constraint description

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