# Notes on the Scenario Design Approach

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***Abstract*—The scenario optimization method developed in [5] is a theoretically sound and practically effective technique for solving in a probabilistic setting robust convex optimization problems arising in systems and control design, that would otherwise be hard to tackle via standard deterministic techniques. In this note, we explore some further aspects of the scenario methodology, and present two results pertaining to the tightness of the sample complexity bounds. We also state a new theorem that enables the user to make a-priori probabilistic claims on the scenario solution, with one level of probability only.**

***Index Terms*—Probabilistic robustness, randomized algorithms, robust control, robust convex optimization, scenario design.**

## I. Preliminaries

Recently, techniques based on uncertainty randomization have gained increasing favor among both control theoreticians and practitioners. Theoreticians are attracted by the solid foundations of these methods, rooting in the theory of probability, optimization and stochastic processes, while practitioners are interested in their relative simplicity of practical implementation. An up-to-date description of this body of techniques, along with applications to control analysis and design problems and many pointers to the literature, can be found in the texts [7], [19]. Among these techniques, the so-called *scenario design* method developed in [5] permits one to effectively solve control design problems that can be cast in the form of a convex optimization program with uncertain constraints. A significant class of control problems indeed fall in this framework, see for instance the discussion and examples in [5]. In what follows, we briefly review the essential points of the scenario optimization approach of [5] in order to prepare the terrain for our further discussion. We also refer the reader to the recent contributions [1], [2], [9], [10] for further information on the scenario approach.

*Scenario Optimization:* Consider an uncertain convex optimization problem of the form

$$\min_{\theta\in\Theta} \quad c^T\theta \quad \text{subject to :} \\ f(\theta,\delta)\le 0,\ \delta\in\Delta \tag{1}$$

where $\theta\in\Theta\subseteq\mathbb{R}^{n_\theta}$ is the decision variable, $\Theta$ is convex and closed, $\delta\in\Delta\subseteq\mathbb{R}^{n_\delta}$ is an uncertain parameter, $c\in\mathbb{R}^{n_\theta}$ is a given vector, and $f(\theta,\delta):\Theta\times\Delta\to[-\infty,\infty]$ is continuous and convex in $\theta$, for any fixed value of $\delta\in\Delta$.

If "Prob" is a probability measure on $\Delta$, the *scenario solution* $\hat\theta_N$ for (1) is the optimal solution of the following convex program

$$\min_{\theta\in\Theta} \quad c^T\theta \quad \text{subject to :} \\ f\left(\theta,\delta^{(i)}\right)\le 0,\ i=1,\dots,N \tag{2}$$

where $\delta^{(i)}$, $i=1,\dots,N$, are independent samples, identically distributed according to Prob. Note that the optimal solution $\hat\theta_N$ of this program is a random variable that depends on the random extractions $\delta^{(1)},\dots,\delta^{(N)}$.

Let $\epsilon\in(0,1)$, $\beta\in(0,1)$ be given (small) probability levels. A key fact in scenario optimization (see equation (12) in [5]) is that if $N$ samples are used in (2), where $N$ is such that

$$\binom{N}{n_\theta}(1-\epsilon)^{N-n_\theta}\le\beta \tag{3}$$

then it holds that

$$\text{Prob}^N\left\{\left(\delta^{(1)},\dots,\delta^{(N)}\right)\in\Delta^N : V(\hat\theta_N)\le\epsilon\right\}\ge 1-\beta \tag{4}$$

where $V(\theta)$ is a measure of violation of the constraints in (1) for a given $\theta$, i.e.

$$V(\theta)\doteq\text{Prob}\left\{\delta\in\Delta : f(\theta,\delta)>0\right\}.$$

In other words, with high probability $1-\beta$, the scenario solution is feasible for all the constraints in (1), except possibly for those in a set having probability measure smaller than $\epsilon$, that is, this solution is "almost robustly feasible." Moreover, Theorem 1 and Corollary 1 in [5] provide explicit expressions $N(\epsilon,\beta)$ (see (5) below) such that if $N\ge N(\epsilon,\beta)$ then (3) and (4) hold. A fundamental point here is that $N(\epsilon,\beta)$ is computed a-priori, before any constraint is extracted, according to the formula[1]

$$N(\epsilon,\beta)=\left\lceil\frac{2}{\epsilon}\ln\frac{1}{\beta}+2n_\theta+\frac{2n_\theta}{\epsilon}\ln\frac{2}{\epsilon}\right\rceil \tag{5}$$

($\lceil\cdot\rceil$ =smallest integer greater than or equal to the argument). This bound holds in full generality for any uncertain convex program, and any probability distribution on the uncertainties. Since $\beta$ in (5) appears under the sign of logarithm, in practice the $\beta$ level can be fixed to a very small value ($10^{-10}$ or even $10^{-20}$) without increasing too much the required number of samples.

*Content of This Note:* In the next sections we provide further results and discussion that are useful for clarifying the scope of the results in [5] and for defining possible margins of improvement. In particular, Section II elaborates on the possibility of improving the dependence on $\epsilon$ and $\beta$ appearing in the sample complexity (5), and provides an essentially negative answer. Section III discusses sequential implementations of the scenario method, and gives a fundamental limit for the expected value of the stopping time of this version of the scenario algorithm. Finally, Section IV gives a new bound for probabilistic assessments on the scenario solution involving a single level of probability.

## II. Dependence on $\epsilon$ and $\beta$ of the Sample Complexity

We show that no general sample complexity bound can be found that scales below $O(\epsilon^{-1}\ln\beta^{-1})$. Comparing with (5), we thus see that the fundamental dependence on $\epsilon$ and $\beta$ appearing in (5) is intrinsic and cannot be improved.

*Proposition 1:* The number $N$ of samples guaranteeing that, in any given problem instance and for any probability distribution, the solution of (2) satisfies (4) must scale at least as

$$O\left(\frac{1}{\epsilon}\ln\frac{1}{\beta}\right).$$

[1]Note that a bound better than (5) is given in Theorem 1 of [5]. For the sake of simplicity, we use here the simplified bound (5) in our discussion.

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*Proof:* A proof is obtained by producing a simple instance of problem (2) in which the minimum $N$ satisfying (4) can be computed exactly, and showing that this number scales as $O(\epsilon^{-1}\ln\beta^{-1})$. To this end, consider the uncertain convex program

$$\min_{\theta\in\mathbb{R}} \quad \theta \quad \text{subject to :}$$
$$\delta - \theta \leq 0, \delta \in [0,1]$$

with a uniform probability distribution on $\Delta = [0,1]$, and the corresponding scenario solution

$$\hat{\theta}_N = \arg\min_{\theta\in\mathbb{R}} \quad \theta \quad \text{subject to :}$$
$$\delta^{(i)} - \theta \leq 0, i = 1,\ldots,N. \tag{6}$$

Clearly, in this case $\hat{\theta}_N = \max_{i=1,\ldots,N}\delta^{(i)}$, and

$$V(\theta) = \text{Prob}\{\delta\in\Delta : \delta - \theta > 0\} = 1-\theta.$$

Therefore, we have that

$$V(\hat{\theta}_N) = 1 - \hat{\theta}_N = 1 - \max_{i=1,\ldots,N}\delta^{(i)}$$

and hence

$$\begin{aligned}
&\text{Prob}^N\left\{\left(\delta^{(1)},\ldots,\delta^{(N)}\right)\in\Delta^N : V(\hat{\theta}_N) > \epsilon\right\}\\
&= \text{Prob}^N\left\{\left(\delta^{(1)},\ldots,\delta^{(N)}\right)\in\Delta^N :\right.\\
&\qquad \left.\max_{i=1,\ldots,N}\delta^{(i)} < 1-\epsilon\right\}\\
&= \text{Prob}^N\left\{\left(\delta^{(1)},\ldots,\delta^{(N)}\right)\in\Delta^N :\right.\\
&\qquad \left.\delta^{(i)} < 1-\epsilon, i = 1,\ldots,N\right\}\\
&= (1-\epsilon)^N.
\end{aligned}$$

It follows that (4) is satisfied with equality, that is

$$\text{Prob}^N\left\{\left(\delta^{(1)},\ldots,\delta^{(N)}\right)\in\Delta^N : V(\hat{\theta}_N)\leq\epsilon\right\} = 1-\beta$$

with $\beta = (1-\epsilon)^N$. Making this formula explicit with respect to $N$ gives $N = \ln\beta^{-1}/\ln(1-\epsilon)^{-1}$. Since $\ln(1-\epsilon)^{-1}$ is convex in [0,1), considering the chord from the origin to point (1/2, $\ln 2$) we have that $\ln(1-\epsilon)^{-1}\leq 2\epsilon\ln 2$, for $\epsilon\in[0,1/2]$; hence

$$N = \frac{\ln\beta^{-1}}{\ln(1-\epsilon)^{-1}} \geq \frac{1}{2\ln 2}\frac{1}{\epsilon}\ln\frac{1}{\beta}, \quad \text{for}\,\epsilon\in[0,1/2].$$

Since this specific scenario problem has a sample complexity that grows at least as $O(\epsilon^{-1}\ln\beta^{-1})$, we have proved that no general sample complexity bound may exist for scenario optimization that scales better than $O(\epsilon^{-1}\ln\beta^{-1})$. □

A couple of remarks are in order.

1) Notice that in the proof of Proposition 1 a simple uncertain linear program with just one variable is produced, whose sample complexity grows indeed as $O(\epsilon^{-1}\ln\beta^{-1})$. The fact that the general bound (5) scales similarly to how it scales in this extremely simple example, and yet the bound applies to all convex problems and all possible distributions, shows that all convex problems share unexpected similarities, as far as sample complexity is concerned.
2) Although Proposition 1 states that there is not much room for improvement upon the a-priori general bound (5), better bounds can still be found for properly modified scenario-like approaches exploiting a-priori knowledge on the structure of uncertainty. For instance, Nemirovski and Shapiro in [14], [15] achieved a $O(\ln\epsilon^{-1}\ln\beta^{-1})$ sample complexity bound by considering a special situation of linear matrix inequality constraints, affinely perturbed by independent parameters having "light-tailed" distributions.

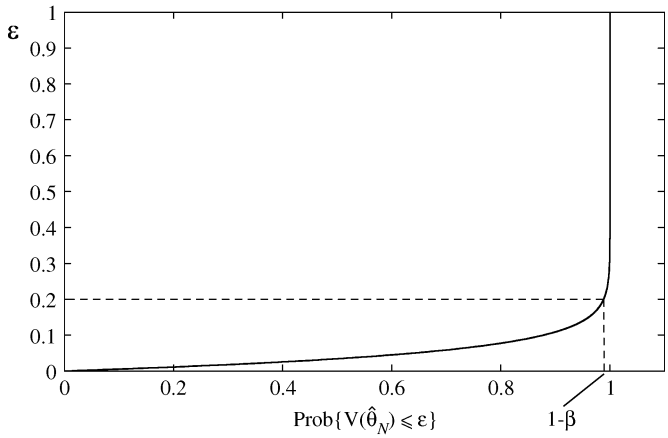


Fig. 1. Allowed violation level $\epsilon$ versus probability for the scenario example (6) with $N = 20$.

## III. A-Priori and On-the-Go Violation

It is worth underlining once more that bound (5) works a-priori, in the sense that the user knows a-priori (i.e. before seeing the actual constraints) that, if $N(\epsilon,\beta)$ samples will be used in the scenario optimization, the resulting solution will satisfy (4).

The violation $V(\hat{\theta}_N)$ is a random variable that depends on the to-be-extracted constraints, and (4) says that this random variable is concentrated around small values. Considering for instance the example in (6), setting $N = 20$, we can plot the a-priori violation level $\epsilon$ versus the probability of the event $\{V(\hat{\theta}_N)\leq\epsilon\}$, see Fig. 1. Note then that, once the constraints have been extracted, the resulting constraint violation can be much lower than the limit $\epsilon$ imposed a-priori. In the example, if we fix a-priori $\epsilon = 0.2$, there is a 0.9885 a-priori probability of achieving a violation smaller than $\epsilon$. The actual violation achieved a-posteriori can however be significantly smaller than 0.2.

This observation might suggest that there could be room for improving the sample complexity, if the number of scenarios is chosen "on-the-go" instead of a-priori.

By "on-the-go" we mean that optimization might be performed by a sequential algorithm that, by checking a current optimal solution obtained on the basis of the samples accrued so far, detects whether the violation is above $\epsilon$ and, in the positive case, extracts a new sample and iterates; otherwise, the algorithm stops and returns the current solution, which guarantees the desired violation level. In other words, the algorithm possesses a stopping-rule to decide when to stop introducing new constraints.

Recently, many sequential randomized methods have been proposed, see [6], [8], [12], [16], [17]. These methods work quite satisfactorily for probabilistic *feasibility* problems, i.e. for design problems where the goal is to find a solution that satisfies (4), whereas, to the best of these authors' knowledge, they still cannot deal satisfactorily with the problem of optimizing an objective subject to (4). We believe that sequential probabilistic methods (here also named "incremental" methods) for *optimization* under uncertainty should be a main research topic for researchers interested in randomized design techniques. Our contribution here limits to define an "abstract" scheme for incremental

optimization, followed by a general result on the best expected value that can possibly be achieved for the stopping-time of such a general incremental method.

*Definition 1: (Incremental Scenario Optimization Scheme):* Define

$$\hat{\theta}_k = \arg\min_{\theta\in\Theta} \quad c^T\theta \quad \text{subject to :} \\ f\left(\theta, \delta^{(i)}\right) \le 0,\ i = 1, \ldots, k$$

Let $k_0 > 0$ be a given integer, and set $k = k_0$.

1) Extract $k$ samples $\delta^{(1)}, \ldots, \delta^{(k)}$;
2) Compute $\hat{\theta}_k$;
3) Check if $V(\hat{\theta}_k) \le \epsilon$. If yes, stop and return the current solution; otherwise set $k = k + 1$, extract a new sample $\delta^{(k)}$, and go to 2.□

Mathematically speaking, the stopping-rule of an incremental scheme is a stopping-time, that is, a discrete random variable that depends on the algorithm history up to the current time, and the sample complexity of an incremental method can be assessed by providing an a-priori bound on its expected value. The following proposition establishes that there exist a fundamental limit to the achievable expected stopping time, and shows that it cannot scale below $O(1/\epsilon)$. We thus find the same dependence on $\epsilon$ we had for a-priori evaluations.

*Proposition 2:* The stopping-time of an incremental optimization scheme applicable to general uncertain convex optimization problems must exhibit at best an expected value that scales as $O(1/\epsilon)$.

*Proof:* A proof of this statement is obtained by considering again the example in (6). Suppose this problem is solved incrementally according to the scheme in Definition 1, with $k_0 = 1$. Define the stopping-time

$$N \doteq \text{number of iterations executed upon exit}$$

and notice that the event $\{N = k\}$ happens if the sequential scheme actually fails to find a good solution for the first $k-1$ iterations, and then it stops at the $k$-th iteration with a good solution. Let us compute the probability of this event (this probability is computed in $\Delta^\infty$, the set of infinite extractions, since the number of extractions is not a-priori defined):

$$\text{Prob}^\infty\{N = k\} = \text{Prob}^k\left\{\left(\delta^{(1)}, \ldots, \delta^{(k)}\right) : \right. \\ \left. V(\hat{\theta}_1) > \epsilon, V(\hat{\theta}_2) > \epsilon, \ldots, V(\hat{\theta}_{k-1}) > \epsilon, V(\hat{\theta}_k) \le \epsilon\right\}. \quad (7)$$

In problem (6) we have that $V(\hat{\theta}_\ell) > \epsilon$ if and only if $\max_{i=1,\ldots,\ell}\delta^{(i)} < (1-\epsilon)$, i.e. if and only if $\delta^{(i)} < (1-\epsilon)$ for $i = 1, \ldots, \ell$. The joint event $\{V(\hat{\theta}_1) > \epsilon, V(\hat{\theta}_2) > \epsilon, \ldots, V(\hat{\theta}_{k-1}) > \epsilon\}$ is thus equivalent to the event $\{\delta^{(i)} < (1-\epsilon), \text{for } i = 1, \ldots, k-1\}$, and the event considered in (7) is the event

$$\left\{\delta^{(i)} < (1-\epsilon),\ \text{for } i = 1, \ldots, k-1,\ \text{and } \delta^{(k)} \ge (1-\epsilon)\right\}.$$

Thus,

$$\text{Prob}^\infty\{N = k\} = \epsilon(1-\epsilon)^{k-1}.$$

We can now compute exactly the expectation for the stopping-time:

$$E\{N\} = \sum_{k=1}^{\infty} k\text{Prob}^\infty\{N = k\} = \sum_{k=1}^{\infty} k\epsilon(1-\epsilon)^{k-1} \\ = \epsilon\sum_{k=1}^{\infty} k(1-\epsilon)^{k-1} = \epsilon\frac{1}{\epsilon^2} = \frac{1}{\epsilon}.$$

Since at least a problem instance exists in which the expected stopping-time grows as $1/\epsilon$, we have proved that no incremental scheme can have an expected stopping-time that scales better than $O(1/\epsilon)$ uniformly over all possible problem instances. □

## IV. Assessments With a Single Level of Probability

In the usual approach to probabilistic robustness, results are given in the form of a statement involving a double level of probability. For instance, (4) states that the probability of violation $V(\hat{\theta}_N)$ is less than or equal to $\epsilon$, with probability at least $1-\beta$. This nested probabilistic statement may cause some confusion to the uninitiated reader. We next show that an alternative result can be obtained for scenario design, which involves one level of probability only.

To make things concrete, let us consider a problem of Lyapunov quadratic state feedback stabilization, and let us ask the following question: *What is the probability that we extract $N$ plants, do a scenario design, and then another plant (the "real" plant) picked at random according to the same probability distribution does not satisfy the designed Lyapunov inequality?* Note that there is only one level of probability here. In the notation of this technical note, this question amounts to assessing the probability with which a scenario solution $\hat{\theta}_N$ (which, we recall, is computed on the basis of the randomly extracted samples $\delta^{(1)}, \ldots, \delta^{(N)}$) fails to satisfy the constraint $f(\hat{\theta}_N, \delta) \le 0$ on a newly extracted $\delta \in \Delta$. That is, we need to evaluate the probability

$$P_B \doteq \text{Prob}^{N+1}\left\{\left(\delta^{(1)}, \ldots, \delta^{(N)}, \delta\right) \in \Delta^N \times \Delta : f(\hat{\theta}_N, \delta) > 0\right\}$$

or, equivalently, the probability

$$P_G \doteq 1 - P_B \\ = \text{Prob}^{N+1}\left\{\left(\delta^{(1)}, \ldots, \delta^{(N)}, \delta\right) \in \Delta^N \times \Delta : \right. \\ \left. f(\hat{\theta}_N, \delta) \le 0\right\}. \quad (8)$$

The following proposition shows that an explicit lower bound for $P_G$ can be determined in full generality.

*Proposition 3:* Consider the scenario solution $\hat{\theta}_N$ of problem (2), with $N \ge n_\theta$. The a-priori probability (8) with which the design inequality $f(\hat{\theta}_N, \delta) \le 0$ is satisfied is

$$P_G \ge \frac{N + 1 - n_\theta}{N+1}. \quad (9)$$

*Proof:* $P_B$ can be rewritten as

$$P_B = \text{Prob}^{N+1}\left\{\left(\delta^{(1)}, \ldots, \delta^{(N)}, \delta\right) \in \Delta^N \times \Delta : \right. \\ \left. f(\hat{\theta}_N, \delta) > 0\right\} \\ = \int_{\Delta^N\times\Delta} \mathbb{I}\left\{f(\hat{\theta}_N, \delta) > 0\right\} \\ \text{dProb}^N\left(\delta^{(1)}, \ldots, \delta^{(N)}\right)\text{dProb}(\delta) \\ = \int_{\Delta^N}\left(\int_\Delta \mathbb{I}\left\{f(\hat{\theta}_N, \delta) > 0\right\}\text{dProb}(\delta)\right) \\ \text{dProb}^N\left(\delta^{(1)}, \ldots, \delta^{(N)}\right).$$

The inner integral in the expression above is nothing but $\text{Prob}\{\delta \in \Delta : f(\hat{\theta}_N, \delta) > 0\}$, which is $V(\hat{\theta}_N)$; hence

$$P_B = \int_{\Delta^N} V(\hat{\theta}_N)\text{dProb}^N\left(\delta^{(1)}, \ldots, \delta^{(N)}\right) \\ = E_{\text{Prob}^N}\left[V(\hat{\theta}_N)\right]. \quad (10)$$

Invoking Theorem 1 in [4], $E_{\text{Prob}^N}[V(\hat{\theta}_N)]$ can be bounded by

$$E_{\text{Prob}^N}\left[V(\hat{\theta}_N)\right] \le \frac{n_\theta}{N+1}$$

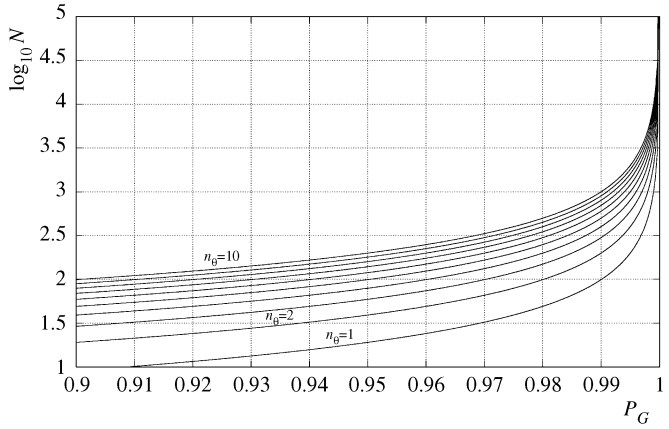


Fig. 2. Logarithmic plot of $N$ satisfying (9) as a function of $P_G$, for $n_\theta = 1, \ldots, 10$.

from which

$$P_G = 1 - P_B = 1 - E_{\mathrm{Prob}^N}\left[V(\hat{\theta}_N)\right] \geq 1 - \frac{n_\theta}{N+1} = \frac{N+1-n_\theta}{N+1}$$

that is the theorem statement. □

It can actually be proved (see [3], [11]) that the result in (9) is tight, since it holds with equality for the so-called "fully-supported" problems. Interestingly, [11] offers a way to establish (9) alternative to the use of Theorem 1 in [4]: Theorem 1 in [11] delivers an exact evaluation of the probability with which $V(\hat{\theta}_N) > \epsilon$, and (9) can also be obtained by integration of this probability. In [3], yet another alternative route is offered to obtain an exact expression of $P_B$.

Formula (9) returns the probability that, if we solve a scenario optimization problem on the basis of $N$ plant samples, the obtained solution is also feasible for another plant extracted according to the same probability. Relation (9) can also be used to design an experiment, where one wishes to a-priori fix a desired level for $P_G$, and then determine the number $N$ of scenarios necessary for achieving this level of probability. For the purpose of illustration, a plot of $N$ as a function of $P_G$ for various values of $n_\theta$ is given in Fig. 2.

### A. A Special Case: Estimation of Extrema via Sampling

A problem that arises frequently in the analysis of robustness of uncertain control systems is that of computing the worst-case value (with respect to the uncertainty) of a function $f(\delta)$ representing some performance or cost index of the system, that is one wants to evaluate the maximum value of $f(\delta)$, for $\delta \in \Delta$. Exact computation of the maximum is in general NP-hard, but randomized techniques may be employed to compute an *estimate* of the maximum. A well known approach (see [13], [18]) is to use the sample maximum as an estimate:

$$\hat{\theta}_N \doteq \max_{i=1,\ldots,N} f\left(\delta^{(i)}\right) \tag{11}$$

where $\delta^{(i)}$, $i = 1, \ldots, N$, are independent, identically distributed samples.

We note that (11) is a special instance of the general problem family (2), where $\theta$ is one-dimensional, and $f(\theta, \delta) \doteq f(\delta) - \theta$. Indeed, problem (11) can be rewritten equivalently in the form (2) as:

$$\hat{\theta}_N \doteq \arg\min_{\theta \in \mathbb{R}} \quad \theta \quad \text{subject to:} \quad f\left(\delta^{(i)}\right) - \theta \leq 0, i = 1, \ldots, N. \tag{12}$$

Proposition 3 gives in this case:

*Proposition 4:* Consider the sample maximum estimate $\hat{\theta}_N$ resulting from problem (12). The a-priori probability (8) with which $f(\delta) \leq \hat{\theta}_N$ holds is

$$P_G \geq \frac{N}{N+1} \tag{13}$$

(relation (13) is actually valid with equality if the distribution of $f(\delta)$ is continuous).

Notice that making explicit (13) with respect to $N$, we have that $P_B \leq \varepsilon$ holds whenever the following sample size bound holds:

$$N \geq \frac{1}{\varepsilon} - 1.$$

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