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ADAPTIVE LQG CONTROL OF INPUT-OUTPUT SYSTEMS—A COST-BIASED APPROACH*

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Abstract. In this paper, we consider linear systems in input-output form and introduce a new adaptive linear quadratic Gaussian (LQG) control scheme which is shown to be self-optimizing. The identification algorithm incorporates a cost-biasing term, which favors the parameters with smaller LQG optimal cost and a second term that aims at moderating the time-variability of the estimate. The corresponding closed-loop scheme is proven to be stable and to achieve an asymptotic LQG cost equal to the one obtained under complete knowledge of the true system (self-optimization).

The results of this paper extend in a nontrivial way previous results established along the cost-biased approach in other settings.

Key words. LQG adaptive control, least squares identification, cost-biased identification, self-optimality

AMS subject classifications. 93E20, 93E15, 93E24, 49L20

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1. Introduction.

Since the appearance of the original contribution of Aström and Wittenmark [1], the analysis of self-tuning control systems has constituted a challenging topic for theorists working in the area of adaptive control. The first significant convergence results were obtained in the late 1970s for minimum-variance control schemes. In particular, a global convergence result for an adaptive control system based on the stochastic gradient algorithm has been established in [13]. Extensions to the least squares (LS) algorithm are dealt with in [32] by introducing a suitable modification to the standard recursive least squares algorithm. Such a modification is in fact not necessary in order to achieve optimality [20].

The common result of all the above-mentioned contributions is that a minimum-variance self-tuning control system obtains under various operating conditions the same performance as the one achievable under complete knowledge of the true plant (*self-optimization*). It is important, however, to emphasize that the minimum-variance control law calls for the restrictive—and often unrealistic—assumption that the plant is minimum-phase. Extending these results to more general control techniques suitable for nonminimum-phase plants has attracted much attention in the last decade. The corresponding analysis, however, is far more complex.

It is by now well known (see, e.g., [21, 22, 18, 25, 33]) that the self-optimization result does not hold true for general control laws based on the minimization of multistep performance indexes. As a matter of fact, the interplay between identification and control in a certainty equivalence adaptive control scheme may result in the convergence of the parameter estimate to a parameterization different from the true one in absence of suitable excitation conditions (see, e.g., [5, 18, 2, 6]). When a cost criterion other than the output variance is considered, this identifiability problem results in a strictly suboptimal performance. In particular, the identifiability problem

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is significant in infinite-horizon linear quadratic Gaussian (LQG) control and, in fact, in [33] it is proven that for a state space system subject to Gaussian noise the set of the parameterizations leading to optimality of LQG control is strictly contained in the set of the potential convergence points.

A first approach to achieve optimality consists of securing the parameter consistency by introducing suitable probing signals in the control system. The probing signals should be sufficiently exciting so that consistency is achieved, and—at the same time—mild enough in order not to degrade the control system performance. In [8, 9, 10, 14, 28], this is obtained by a careful selection of an asymptotically vanishing dither noise. This approach is useful only in the case when noise injection is feasible.

A second approach—adopted in this paper—is based on the so-called *cost-biased method* originally introduced in [21]. In order to better focus on the basic idea underlying this approach and to highlight the main contributions given in the present paper, we proceed as follows: first we introduce the dynamic systems we consider; then we outline the cost-biased approach with specific reference to our class of systems; finally we put our results into perspective with the other existing results obtained along the cost-biased approach.

We consider dynamic systems in input-output form described by the following equation:

$$\mathcal{A}(\vartheta^\circ; q^{-1})\, y_t = \mathcal{B}(\vartheta^\circ; q^{-1})\, u_{t-1} + n_t, \tag{1.1}$$

where $\mathcal{A}(\vartheta^\circ; q^{-1}) = 1 - \sum_{i=1}^{n} a_i^\circ q^{-i}$ and $\mathcal{B}(\vartheta^\circ; q^{-1}) = \sum_{i=1}^{m} b_i^\circ q^{-i+1}$ are polynomials in the unit-delay operator q^{-1} and $\vartheta^\circ = [\, a_1^\circ \; a_2^\circ \ldots a_n^\circ \; b_1^\circ \; b_2^\circ \ldots b_m^\circ \,]^T$ is the system parameter vector. The control objective is to minimize the quadratic cost

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} [\, y_t^2 + \beta\, u_t^2], \tag{1.2}$$

where the control weighting coefficient β is strictly positive.

The basic idea of the cost-biased approach can be outlined as follows.

Suppose a standard LS algorithm is used for the identification of system (1.1) and let $\hat{\vartheta}_t^{LS}$ be the corresponding LS estimate at time t. According to the certainty equivalence principle, the control action is obtained by the relation $u_t = u_t^\star(\hat{\vartheta}_t^{LS})$, where $u_t^\star(\vartheta)$ indicates the optimal LQG control law for system (1.1) with parameter ϑ. For ease of reference, let us introduce the symbol $S(\vartheta_1, \vartheta_2)$ for the control system formed by system (1.1) with parameter ϑ_1 with the loop closed by $u_t = u_t^\star(\vartheta_2)$. Since the identification is performed in closed-loop, it is expected that the behavior of $S(\vartheta^\circ, \hat{\vartheta}_t^{LS})$ will be the same, at least in the long run, as the one of $S(\hat{\vartheta}_t^{LS}, \hat{\vartheta}_t^{LS})$. Then, the LQG cost for $S(\vartheta^\circ, \hat{\vartheta}_t^{LS})$—i.e., the incurred cost—will be the same as the LQG cost for $S(\hat{\vartheta}_t^{LS}, \hat{\vartheta}_t^{LS})$. However, one should note that the latter configuration is optimal for the estimated model, whereas the incurred cost obviously cannot be lower than the optimal cost for the true system. From this, one concludes that the least squares algorithm has a natural tendency to return estimates with an optimal cost that is not smaller than the optimal cost associated with the true system and that, when it is strictly larger, the adaptive scheme attains a suboptimal performance.

In the cost-biased approach an extra term that favors parameters with smaller optimal cost is added to the LS identification cost. This extra term is selected with a twofold objective. On the one hand, it should be strong enough so that the optimal LQG cost associated with the estimated model is asymptotically not larger than the optimal cost for the true system. If, on the other hand, it is so mild that the closed-loop identification property $S(\vartheta^\circ, \hat{\vartheta}_t^{LS}) = S(\hat{\vartheta}_t^{LS}, \hat{\vartheta}_t^{LS})$ is preserved, then one still has that the incurred cost is equal to the cost for $S(\hat{\vartheta}_t^{LS}, \hat{\vartheta}_t^{LS})$. From this, optimality of the adaptive control scheme is achieved.

The cost-biased approach has been successfully applied in a number of different settings. Controlled Markov chains with a finite parameter set are considered in [21]. The results of this paper have been extended to Markov chains with an infinite parameter set in [24] and to systems with a general state space but still with a finite parameter set in [19].

Linear systems in a state space representation are dealt with in [18] and [7]. In these papers, the restrictive assumption that the state is fully accessible is made. Moreover, it is assumed that the noise system affects all state variables. This assumption is crucial for the correct functioning of the proposed identification procedure. As a matter of fact, the presence of a full-range noise sheds light on the existing difference between the true system and the estimated model and this helps the identification task. In the paper [7], it is in fact shown that this mechanism is effective enough so as to counteract the biasing effect of the cost-biasing term thus guaranteeing the closed-loop identification property. Unfortunately, the assumption that the noise is full-range is so restrictive that it cannot be applied to many situations of interest. In particular, a state space realization of the input-output system (1.1) does not satisfy this condition.

In the present paper, an optimal adaptive control scheme for system (1.1) still based on the cost-biasing idea is presented. Extending the cost-biased approach to systems as (1.1) is important in that input-output systems are largely used in adaptive control applications. Moreover, assuming only the input and output measurability is much more realistic than assuming full state accessibility. As a side remark we also note that, in contrast with [18] and [7], our approach does not require the noise to be Gaussian.

The paper is organized as follows: in section 2, we describe the cost-biased adaptive LQG control scheme and recall some relevant properties of the standard LS estimates. The study of the cost-biased identification algorithm is presented in section 3. Section 4 is devoted to the analysis of the closed-loop stability and the characterization of the self-tuning LQG control performance. Finally, section 5 presents conclusions and suggestions for future research.

2. The cost-biased adaptive LQG control system.

2.1. The LQG optimal control problem. In this section, we summarize some known facts on infinite-horizon LQG control relevant for the subsequent developments. This is also useful in order to introduce the assumptions and the notations we shall use throughout the paper.

Consider the discrete time single input, single output (SISO) system (1.1) where signal n_t is a stochastic disturbance precisely described in the following.

ASSUMPTION 2.1. *$\{n_t\}$ is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$, satisfying the following conditions:*

1. $\sup_t E[|n_t|^p/\mathcal{F}_{t-1}] < \infty$ *almost surely (a.s)* $\forall p > 0$*;*
2. $\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} n_t^2 = \sigma^2 > 0$ *a.s.*

Note that Assumption 2.1 is satisfied when $\{n_t\}$ is an independently and identically distributed (i.i.d.) Gaussian sequence, but it includes many other situations.

We make the assumption on system (1.1) that $n > 0$ (nontrivial autoregressive part). Note that if $n = 0$, the trivial control law $u_t = 0$, $t \geq 0$, is obviously optimal irrespective of the value of ϑ°.

We further assume that system (1.1) belongs to a known set of stabilizable models according to the following.

ASSUMPTION 2.2. *$\vartheta^\circ \in \Theta$, where Θ is a compact set such that $\Theta \subset \{\vartheta \in \Re^{n+m} : q^s\mathcal{A}(\vartheta; q^{-1})$ and $q^{s-1}\mathcal{B}(\vartheta; q^{-1})$ have no unstable pole-zero cancellations$\}$, $s = \max\{n, m\}$ being the order of the system.*

System (1.1) is initialized at time $t = 0$ with $y_t = u_{t-1} = 0$, $t \leq 0$.

For the determination of an optimal control law, it is convenient to represent system (1.1) in a state space form such that the state is accessible and then apply the well-known solution to the optimal LQG control problem for full state accessible state space systems (see, e.g., [10], [3]).

Defining $x_t := [y_t\; y_{t-1} \ldots y_{t-(n-1)}\; u_{t-1}\; u_{t-2} \ldots u_{t-(m-1)}]^T$, system (1.1) can be given the following state space representation of order $\bar{s} := n + m - 1$

$$\begin{cases} x_{t+1} = A(\vartheta^\circ)x_t + B(\vartheta^\circ)u_t + Cn_{t+1}, \quad x_0 = [0\,0 \ldots 0]^T, \\ y_t = Hx_t \end{cases} \tag{2.1}$$

with matrices

$$A(\vartheta) = \left[\begin{array}{cccc|cccc} a_1 & \ldots & a_{n-1} & a_n & b_2 & \ldots & b_{m-1} & b_m \\ 1 & 0 & \ldots & & 0 & \ldots & & 0 \\ & \ddots & \ddots & & & \ddots & & 0 \\ & & 1 & 0 & & & & 0 \\ \hline 0 & \ldots & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 & 1 & 0 & & \\ & \ddots & \ddots & & & \ddots & \ddots & \\ & & 0 & 0 & & & 1 & 0 \end{array}\right],$$

$$B(\vartheta) = \left[\begin{array}{c} b_1 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \\ 0 \\ \vdots \\ 0 \end{array}\right], \quad C = \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right], \quad H = \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right].$$

In this way, the LQG regulation problem for the system in input-output representation (1.1) is reformulated as a complete state information control problem where the performance index to be minimized is given by $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1}[x_t^T T x_t + \beta u_t^2]$, where $T = H^T H \geq 0$ and $\beta > 0$.

Note that, in the case when $n > 1$ and $m > 1$, the state space representation (2.1) of system (1.1) is nonminimal (the order of system (1.1) is $s = \max\{n, m\}$, whereas the dimension of matrix $A(\vartheta^\circ)$ is $\bar{s} = n + m - 1$). However, from the block triangular matrix structure of $A(\vartheta^\circ)$ it is easily seen that the added eigenvalues are identically equal to zero. Hence from Assumption 2.2 it follows that $(A(\vartheta^\circ), B(\vartheta^\circ))$

is stabilizable and $(A(\vartheta^\circ), H)$ is detectable and the standard approach based on the solution to a Riccati equation can be used to determine the control law.

Specifically, the solution to the original LQG control problem has the following expression [10]:

$$u_t = \mathcal{S}(\vartheta^\circ; q^{-1})\, y_t + \mathcal{R}(\vartheta^\circ; q^{-1})\, u_t, \tag{2.2}$$

where $\mathcal{S}(\vartheta^\circ; q^{-1}) = \sum_{i=0}^{n-1} s_i(\vartheta^\circ) q^{-i}$ and $\mathcal{R}(\vartheta^\circ; q^{-1}) = \sum_{i=1}^{m-1} r_i(\vartheta^\circ) q^{-i}$, and coefficients $\{s_i(\vartheta^\circ)\}$ and $\{r_i(\vartheta^\circ)\}$ are computed as follows.

Set $L(\vartheta^\circ) := [\, s_0(\vartheta^\circ)\; s_1(\vartheta^\circ) \dots s_{n-1}(\vartheta^\circ)\; r_1(\vartheta^\circ) \dots r_{m-1}(\vartheta^\circ)\,]$. Then

$$L(\vartheta^\circ) = -(B(\vartheta^\circ)^T P(\vartheta^\circ) B(\vartheta^\circ) + \beta)^{-1} B(\vartheta^\circ)^T P(\vartheta^\circ) A(\vartheta^\circ), \tag{2.3}$$

where $P(\vartheta^\circ)$ is the unique positive semidefinite solution to the discrete time algebraic Riccati equation

$$P = A(\vartheta^\circ)^T \left[P - PB(\vartheta^\circ)(B(\vartheta^\circ)^T PB(\vartheta^\circ) + \beta)^{-1} B(\vartheta^\circ)^T P \right] A(\vartheta^\circ) + H^T H.$$

Moreover, the optimal LQG cost is given by $J^\star(\vartheta^\circ) = \sigma^2 trace(P(\vartheta^\circ) CC^T)$, a.s.

REMARK 2.3. *Since the positive semidefinite solution $P(\vartheta)$ to*

$$P = A(\vartheta)^T \left[P - PB(\vartheta)(B(\vartheta)^T PB(\vartheta) + \beta)^{-1} B(\vartheta)^T P \right] A(\vartheta) + H^T H \tag{2.4}$$

is analytic as a function of the parameter vector ϑ in the set $\mathcal{C} = \{\vartheta \in \Re^{n+m} : q^s \mathcal{A}(\vartheta; q^{-1})$ and $q^{s-1}\mathcal{B}(\vartheta; q^{-1})$ have no unstable pole-zero cancellations$\}$ (see [12]*), it is easily seen that $s_i(\vartheta)$, $r_i(\vartheta)$, and $J^\star(\vartheta)$ are analytic functions of ϑ, $\vartheta \in \mathcal{C}$, as well.*

2.2. The cost-biased identification algorithm. Introducing the observation vector $\varphi_t := [\, y_t \dots y_{t-(n-1)}\; u_t \dots u_{t-(m-1)}]^T$, system (1.1) can be given the regression-like form

$$y_t = \varphi_{t-1}^T \vartheta^\circ + n_t, \tag{2.5}$$

and the LS identification index for the estimate of ϑ° is [26]

$$V_t(\vartheta) = \sum_{s=1}^{t} (y_s - \varphi_{s-1}^T \vartheta)^2. \tag{2.6}$$

In the theorem below, we recall a fundamental result for the LS estimate proven in [23, Theorem 1].

THEOREM 2.4. *Suppose that u_t is $\mathcal{F}_t$-measurable. Then*

$$(\vartheta^\circ - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta^\circ - \hat{\vartheta}_t^{LS}) = O\left(\log \lambda_{max} \left(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T \right) \right) \quad a.s. \tag{2.7}$$

In particular, this implies that under the conditions

(i) $\lambda_{min} \left(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T \right) \to \infty$ *a.s.,*

(ii) $\log \lambda_{max} \left(\sum_{s=0}^{t} \varphi_{s-1}\varphi_{s-1}^T \right) = o\left(\lambda_{min} \left(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T \right) \right)$ *a.s.,* *the LS estimate is consistent.*

In adaptive control, identification is performed in closed-loop. Therefore, one cannot ensure the satisfaction of conditions (i) and (ii) and the true parameter vector is generally not consistently estimated. Nevertheless, property (2.7) still provides a valuable bound on the discrepancy between the estimated parameter and the true parameter. We call this property "closed-loop identification property" to emphasize that it holds even in closed-loop. On the other hand, as discussed in section 1, the LS identification algorithm generally provides estimates with an optimal LQG cost larger than the optimal cost associated with the true system. This is the reason why optimality of an LS-based adaptive control scheme is not guaranteed.

Motivated by these considerations, we introduce a cost-biased identification algorithm with the twofold objective of preserving the LS property (2.7) and forcing the estimates to lie asymptotically in the parameter region with an optimal cost not larger than the optimal cost associated with the true system.

Consider the estimate $\hat{\vartheta}_t$ computed through the following algorithm:

$$\hat{\vartheta}_t = \begin{cases} \arg\min_{\vartheta \in \Theta} D_t(\vartheta) & \text{if } t = t_i,\ i = 0, 1, 2, \ldots, \\ \hat{\vartheta}_{t-1} & \text{otherwise,} \end{cases} \tag{2.8}$$

where the time instants $\{t_i\}$ are obtained by the recursive equation $t_{i+1} = t_i + T_i$ initialized with $t_0 = 0$ and the cost-biased identification index $D_t(\vartheta)$ is given by

$$D_t(\vartheta) = V_t(\vartheta) + \alpha_t J^\star(\vartheta) + \gamma_t \|\vartheta - \hat{\vartheta}_{t-1}\|,\ \hat{\vartheta}_{-1} = 0, \tag{2.9}$$

where $V_t(\vartheta)$ is the LS cost (2.6) and $J^\star(\vartheta)$ is the optimal LQG cost for system (1.1) with parameter ϑ. The identification algorithm is completely defined by specifying the sequences of

- freezing time intervals $\{T_i\}$,
- cost-biasing weights $\{\alpha_t\}$,
- friction parameters $\{\gamma_t\}$.

We discuss hereafter the meaning of these parameters, while their actual choice is postponed to the following section.

The freezing parameter T_i is used to ensure stability of the closed-loop system. Since the parameter estimate changes with time and the control law is tuned to such an estimate, the adaptive control system is time-varying. On the other hand, it is well known that guaranteeing a stability property at each time instant for the "frozen dynamics" does not imply that the overall time-varying system has a stable dynamics. This problem can be solved by updating the estimate at a slower rate than the updating of the system variables, and this is achieved by a suitable choice of T_i. This same approach is exploited, for instance, in [17], [27], and [29].

The cost-biasing term $\alpha_t J^\star(\vartheta)$ is introduced with the objective of penalizing those parameterizations with high optimal LQG cost. The weight α_t has to be appropriately selected so as to balance the contrasting objectives of preserving the closed-loop identification property (2.7) and forcing the asymptotic estimate to correspond to a model with value of the optimal LQG performance index not larger than the optimal performance value for the true system.

Finally, the friction term $\gamma_t \|\vartheta - \hat{\vartheta}_{t-1}\|$ is introduced so as to avoid the estimate $\hat{\vartheta}_t$ being subject to undesired jumps in the time instants t_i when it is updated. This is necessary to prove optimality of the adaptive scheme.

3. Selection of $\{T_i\}$, $\{\alpha_t\}$, $\{\gamma_t\}$ and properties of $\hat{\vartheta}_t$. The adaptive control law is given by the optimal control law (2.2) with the estimate $\hat{\vartheta}_t$ in place of ϑ° (certainty equivalence principle):

$$u_t = \mathcal{S}(\hat{\vartheta}_t; q^{-1})\, y_t + \mathcal{R}(\hat{\vartheta}_t; q^{-1})\, u_t.$$

The system

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\hat{\vartheta}_t; q^{-1})]\, y_{t+1} + \mathcal{B}(\hat{\vartheta}_t; q^{-1})\, u_t, \\ u_t = \mathcal{S}(\hat{\vartheta}_t; q^{-1})\, y_t + \mathcal{R}(\hat{\vartheta}_t; q^{-1})\, u_t \end{cases} \tag{3.1}$$

is then given the name of *time-varying estimated system*. We will select T_i so as to stabilize system (3.1) and later on in section 4 we shall see that this leads to the stability of the true closed-loop system. Letting $x_t := [y_t \dots y_{t-(n-1)} u_{t-1} \dots u_{t-(m-1)}]^T$, this system can be given the state space representation

$$x_{t+1} = F(\hat{\vartheta}_t)\, x_t \tag{3.2}$$

with

$$F(\vartheta) = A(\vartheta) + B(\vartheta)L(\vartheta), \tag{3.3}$$

where matrices $A(\vartheta)$, $B(\vartheta)$, and $L(\vartheta)$ have been introduced in section 2.1.

Choose now a constant $\mu < 1$ (*contraction* constant). The time interval T_i is then defined as

$$T_i := \inf\{\tau \in Z_+ : \|F(\hat{\vartheta}_{t_i})^\tau\| \leq \mu\} \tag{3.4}$$

(note that such a T_i exists since $\hat{\vartheta}_{t_i}$ belongs to Θ and therefore corresponds to a stabilizable system). In this way, the time-varying system (3.1) is kept constant until its dynamics is contracted by a factor μ, whence guaranteeing its stability. The following proposition makes this precise.

PROPOSITION 3.1. *The autonomous estimated system $x_{t+1} = F(\hat{\vartheta}_t)\, x_t$ is a.s. exponentially stable, uniformly in time.*

Proof. The proof is given in the appendix. □

The choice of $\{\alpha_t\}$ and $\{\gamma_t\}$ is discussed in the next theorem.

THEOREM 3.2. *Suppose that u_t is $\mathcal{F}_t$-measurable. Given $\delta > 0$, select*

$$\alpha_t := \log^{1+\delta} \lambda_{max} \left(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T \right) \tag{3.5}$$

and $\{\gamma_t\}$ to be a positive diverging sequence of real numbers satisfying $\gamma_t = o(\alpha_t)$. Then,

(i) $(\vartheta^\circ - \hat{\vartheta}_{t_i})^T \sum_{s=1}^{t_i} \varphi_{s-1}\varphi_{s-1}^T (\vartheta^\circ - \hat{\vartheta}_{t_i}) = O\left(\log^{1+\delta} \lambda_{max} \left(\sum_{s=1}^{t_i} \varphi_{s-1}\varphi_{s-1}^T \right) \right)$ *a.s.*,

(ii) $\limsup_{t\to\infty} J^\star(\hat{\vartheta}_t) \leq J^\star(\vartheta^\circ)$ *a.s.*,

(iii) *if* $\sum_{t=1}^{N} \|\varphi_{t-1}\|^2 = O(N)$ *a.s., then* $\sum_{t=1}^{N} \|\hat{\vartheta}_t - \hat{\vartheta}_{t-1}\| = o(N)$ *a.s.*

Proof. The proof is given in the appendix. ∎

According to (3.5), $\{\alpha_t\}$ is chosen to be an increasing sequence of real numbers adaptively selected on the basis of the data generated by the controlled system. According to result (ii), this selection is effective in pushing the estimate towards the region where the optimal LQG cost is not larger than $J^\star(\vartheta^\circ)$. In turn, result (i) shows that the closed-loop identification property (2.7) is preserved with two slight differences: (1) the exponent $1+\delta$ appears in the right-hand side, (2) the rate of divergence in point (i) of Theorem 3.2 concerns only the time instants t_i when the estimate $\hat{\vartheta}_t$ is updated, while the original closed-loop identification property refers to all t's.

Before ending this section, we state a proposition regarding the estimation error

$$e_t := \varphi_t^T[\vartheta^\circ - \hat{\vartheta}_t]. \tag{3.6}$$

The technical proof of this proposition is given in the appendix and is obtained by a suitable manipulation of the sole result (i) in Theorem 3.2.

PROPOSITION 3.3. *The estimation error $e_t = \varphi_t^T[\vartheta^\circ - \hat{\vartheta}_t]$ satisfies the following equation:*

$$\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} |e_t|^p = o\left(\sum_{t=0}^{N} \|\varphi_t\|^p + N\right), \quad p \geq 2,\ a.s.,$$

where $\mathcal{B}_N$ is a set of instant points which depends on N, whose cardinality is bounded: $|\mathcal{B}_N| \leq C_{\mathcal{B}}\ \forall N$.

4. Stability and optimality. The closed-loop system

$$\left\{\begin{array}{l} y_{t+1} = [1-\mathcal{A}(\vartheta^\circ;q^{-1})]\,y_{t+1} + \mathcal{B}(\vartheta^\circ;q^{-1})\,u_t + n_{t+1}, \\ u_t = \mathcal{S}(\hat{\vartheta}_t;q^{-1})\,y_t + \mathcal{R}(\hat{\vartheta}_t;q^{-1})\,u_t \end{array}\right. \tag{4.1}$$

can be represented as a variation system with respect to the so-called estimated system of (3.1) as follows:

$$\left\{\begin{array}{l} y_{t+1} = [1-\mathcal{A}(\hat{\vartheta}_t;q^{-1})]\,y_{t+1} + \mathcal{B}(\hat{\vartheta}_t;q^{-1})\,u_t + n_{t+1} + e_t, \\ u_t = \mathcal{S}(\hat{\vartheta}_t;q^{-1})\,y_t + \mathcal{R}(\hat{\vartheta}_t;q^{-1})\,u_t, \end{array}\right. \tag{4.2}$$

where e_t is defined in (3.6). The uniform stability property of the estimated system (3.1) (Proposition 3.1) and the property of e_t stated in Proposition 3.3 are exploited in the next theorem to prove stability of system (4.1).

THEOREM 4.1 (L^p-stability). *The adaptive LQG control scheme*

$$\left\{\begin{array}{l} y_{t+1} = [1-\mathcal{A}(\vartheta^\circ;q^{-1})]\,y_{t+1} + \mathcal{B}(\vartheta^\circ;q^{-1})\,u_t + n_{t+1}, \\ u_t = \mathcal{S}(\hat{\vartheta}_t;q^{-1})\,y_t + \mathcal{R}(\hat{\vartheta}_t;q^{-1})\,u_t \end{array}\right.$$

is L^p-stable: $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1}[|y_t|^p + |u_t|^p] < \infty$ *a.s.* $\forall p > 0$.

Proof. Fix a time point $N > 0$ and an integer $d \geq 1$.

From Proposition 3.3, there exists a set of instant points $\mathcal{B}_{N-1}$ such that

$$\sum_{t=0,\ t\notin\mathcal{B}_{N-1}}^{N-1} e_t^{2^d} = o\left(\sum_{t=0}^{N-1} \|\varphi_t\|^{2^d} + N\right) \quad \text{a.s.} \tag{4.3}$$

In view of representation (4.2) of system (4.1), it is easily seen that the state vector

$$x_t = [y_t \dots y_{t-(n-1)}\, u_{t-1} \dots u_{t-(m-1)}]^T$$

is governed by the equation

$$x_{t+1} = F^\circ(\hat{\vartheta}_t)\, x_t + C n_{t+1} \tag{4.4}$$

$$= F(\hat{\vartheta}_t)\, x_t + C[e_t + n_{t+1}], \tag{4.5}$$

where $F^\circ(\vartheta) = A(\vartheta^\circ) + B(\vartheta^\circ)L(\vartheta)$, $A(\vartheta^\circ)$, $B(\vartheta^\circ)$, $L(\vartheta)$, and C are defined in section 2.1, and $F(\vartheta)$ is given in (3.3).

For the following derivations, it is convenient to use representation (4.4) in the time instants $t \in \mathcal{B}_{N-1}$ and representation (4.5) for $t \notin \mathcal{B}_{N-1}$, thus finally leading to

$$x_{t+1} = \begin{cases} F^\circ(\hat{\vartheta}_t)\, x_t + C n_{t+1}, & t \in \mathcal{B}_{N-1}, \\ F(\hat{\vartheta}_t)\, x_t + C[e_t + n_{t+1}], & t \notin \mathcal{B}_{N-1}. \end{cases} \tag{4.6}$$

Note now that since $\hat{\vartheta}_t$ belongs to the compact set Θ and $F^\circ(\vartheta)$ is a continuous function of ϑ, $\vartheta \in \Theta$, we then have that $\|F^\circ(\hat{\vartheta}_t)\|$ is uniformly bounded. From this fact and the uniform exponential stability of the autonomous system $x_{t+1} = F(\hat{\vartheta}_t)x_t$ (Proposition 3.1), and the fact that $|\mathcal{B}_{N-1}| \le C_{\mathcal{B}}$ $\forall N$ (see Proposition 3.3), it is easy to show that the state vector x_t generated by system (4.6) can be bounded as follows:

$$\|x_t\| \le k_1 \left\{ \sum_{i=1}^{t} \nu^{t-i}|n_i| + \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i}|e_i| \right\}, \quad t \le N,$$

where k_1 and $\nu \in (0,1)$ are suitable constants. We now have

$$\begin{aligned}
&\left[k_1 \left\{ \sum_{i=1}^{t} \nu^{t-i}|n_i| + \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i}|e_i| \right\} \right]^{2^d} \\
&\quad \le k_1^{2^d} \left[\left\{ \sum_{i=1}^{t} \nu^{t-i}|n_i| + \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i}|e_i| \right\}^2 \right]^{2^{d-1}} \\
&\quad \le k_1^{2^d} \left[2\left\{ \sum_{i=1}^{t} \nu^{\frac{t-i}{2}} (\nu^{\frac{t-i}{2}}|n_i|) \right\}^2 + 2\left\{ \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{\frac{t-i}{2}} (\nu^{\frac{t-i}{2}}|e_i|) \right\}^2 \right]^{2^{d-1}} \\
&\quad \le k_1^{2^d} \left[2\sum_{i=1}^{t} \nu^{t-i} \sum_{i=1}^{t} \nu^{t-i} n_i^2 + 2 \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i} \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i} e_i^2 \right]^{2^{d-1}} \\
&\quad \le k_1^{2^d} \left(\frac{2}{1-\nu} \right)^{2^{d-1}} \left[\sum_{i=1}^{t} \nu^{t-i} n_i^2 + \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i} e_i^2 \right]^{2^{d-1}}.
\end{aligned}$$

Iterating this same equation d times, we then obtain

$$\|x_t\|^{2^d} \le k_2 \left\{ \sum_{i=1}^{t} \nu^{t-i} n_i^{2^d} + \sum_{i=0, i\notin\mathcal{B}_{N-1}}^{t-1} \nu^{t-i} e_i^{2^d} \right\}, \quad t \le N, \tag{4.7}$$

k_2 being a suitable constant, from which we finally get

$$\frac{1}{N}\sum_{t=1}^{N}\|x_t\|^{2^d} \le k_3 \left\{ \frac{1}{N}\sum_{t=1}^{N} n_t^{2^d} + \frac{1}{N}\sum_{t=0, t\notin\mathcal{B}_{N-1}}^{N-1} e_t^{2^d} \right\}, \tag{4.8}$$

where k_3 is a suitable constant, independent of N.

We next bound the two terms in the right-hand side of (4.8).

The term $\frac{1}{N}\sum_{t=1}^{N} n_t^{2^d}$ is handled as follows. Define $v_t := n_t^{2^d} - E[n_t^{2^d}|\mathcal{F}_{t-1}]$. Then $\{v_t\}$ is a martingale difference satisfying

$$\sum_{t=1}^{\infty}\frac{1}{t^2}E[v_t^2|\mathcal{F}_{t-1}] \le \sum_{t=1}^{\infty}\frac{1}{t^2}E[n_t^{2^{d+1}}|\mathcal{F}_{t-1}] < \infty,$$

due to Assumption 2.1. By applying Theorem 2.18 in [16], we then conclude that $\frac{1}{N}\sum_{t=0}^{N-1}[n_t^{2^d} - E[n_t^{2^d}|\mathcal{F}_{t-1}]]$ tends to zero, a.s. Since $\frac{1}{N}\sum_{t=0}^{N-1} E[n_t^{2^d}|\mathcal{F}_{t-1}]$ is bounded by Assumption 2.1, we finally have $\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1} n_t^{2^d} < \infty$, a.s.

The term $\frac{1}{N}\sum_{t=0,t\notin\mathcal{B}_{N-1}}^{N-1} e_t^{2^d}$ is immediately bounded by means of (4.3) and the final bound for $\frac{1}{N}\sum_{t=1}^{N}\|x_t\|^{2^d}$ is obtained

$$\frac{1}{N}\sum_{t=1}^{N}\|x_t\|^{2^d} = O(1) + o\left(\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^{2^d}\right) \quad \text{a.s.}$$

Since $\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^{2^d} \le \frac{1}{N}\sum_{t=0}^{N}\|x_t\|^{2^d}$, this implies that $\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^{2^d}$ remains bounded. Then, the thesis immediately follows from the arbitrariness of d and the fact that $\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^p \le \frac{1}{N}\sum_{t=0}^{N-1}[\|\varphi_t\|^{2^d}+1]$, $p \le 2^d$. ∎

In the next theorem we show that the LQG adaptive control scheme is self-optimizing.

THEOREM 4.2 (optimality). *The adaptive LQG control scheme*

$$\begin{cases} y_{t+1} = [1-\mathcal{A}(\vartheta^\circ;q^{-1})]\,y_{t+1} + \mathcal{B}(\vartheta^\circ;q^{-1})\,u_t + n_{t+1}, \\ u_t = \mathcal{S}(\hat{\vartheta}_t;q^{-1})\,y_t + \mathcal{R}(\hat{\vartheta}_t;q^{-1})\,u_t \end{cases}$$

is self-optimizing: $\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\left[y_t^2 + \beta\,u_t^2\right] = J^\star(\vartheta^\circ)$ a.s.

Proof. We start by showing that $x_t := [y_t \ldots y_{t-(n-1)}\; u_{t-1}\ldots u_{t-(m-1)}]^T$ satisfies the following equation:

$$\|x_t\|^p = o(t), \quad \forall p > 0, \quad \text{a.s.} \tag{4.9}$$

This condition will be useful in the subsequent derivations. By contradiction, suppose that there exist $\{t_k\}_{k\ge0}$ and a real number $\eta > 0$, such that $\|x_{t_k}\| > \eta t_k$ $\forall k$. Then

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\|x_t\|^{1+p} \ge \limsup_{k\to\infty}\frac{1}{t_k}\|x_{t_k}\|^{1+p} \ge \limsup_{k\to\infty}\frac{1}{t_k}\eta^{1+p}t_k^{1+p} = \infty,$$

which contradicts Theorem 4.1.

Observe now that the dynamic programming equation for the estimated model $x_{t+1} = A(\hat{\vartheta}_t)x_t + B(\hat{\vartheta}_t)u_t + Cn_{t+1}$ writes

$$\begin{aligned} J^\star(\hat{\vartheta}_t) + x_t^T P(\hat{\vartheta}_t)x_t = x_t^T T x_t + \beta u_t^2 + E\big[&(A(\hat{\vartheta}_t)x_t + B(\hat{\vartheta}_t)u_t + Cn_{t+1})^T \\ &P(\hat{\vartheta}_t)(A(\hat{\vartheta}_t)x_t + B(\hat{\vartheta}_t)u_t + Cn_{t+1})\,|\,\mathcal{F}_t\big], \end{aligned} \tag{4.10}$$

where $P(\vartheta)$ is the solution to the Riccati equation (2.4). By (2.1) and the definition (3.6), x_t can be given the following expression:

$$x_{t+1} = A(\hat{\vartheta}_t)x_t + B(\hat{\vartheta}_t)u_t + Cn_{t+1} + Ce_t. \tag{4.11}$$

Substituting (4.11) in (4.10), we then get

$$J^\star(\hat{\vartheta}_t) + x_t^T P(\hat{\vartheta}_t)x_t = x_t^T T x_t + \beta u_t^2 + E\big[(x_{t+1} - Ce_t)^T P(\hat{\vartheta}_t)(x_{t+1} - Ce_t)\,|\,\mathcal{F}_t\big],$$

from which

$$\begin{aligned}\frac{1}{N}\sum_{t=0}^{N-1} J^\star(\hat{\vartheta}_t) - \frac{1}{N}\sum_{t=0}^{N-1}[x_t^T T x_t + \beta u_t^2]&\\
= -\frac{1}{N}\sum_{t=0}^{N-1}\Big[x_t^T P(\hat{\vartheta}_t)x_t - E\big[x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1}\,|\,\mathcal{F}_t\big]\Big]&\\
+\frac{1}{N}\sum_{t=0}^{N-1} E\big[x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1}\,|\,\mathcal{F}_t\big]&\\
+\frac{1}{N}\sum_{t=0}^{N-1} E\big[e_t^T C^T P(\hat{\vartheta}_t)Ce_t\,|\,\mathcal{F}_t\big]&\\
-2\frac{1}{N}\sum_{t=0}^{N-1} E\big[x_{t+1}^T P(\hat{\vartheta}_t)Ce_t\,|\,\mathcal{F}_t\big].&\end{aligned} \tag{4.12}$$

From property (ii) in Theorem 3.2, we get $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} J^\star(\hat{\vartheta}_t) \le J^\star(\vartheta^\circ)$ a.s. Therefore, the thesis will be proved if we show that all the terms in the right-hand side of (4.12) tend to zeros as $N \to \infty$. We shall study each term separately.

First term:

$$\begin{aligned}\frac{1}{N}\sum_{t=0}^{N-1}\Big[x_t^T P(\hat{\vartheta}_t)x_t - E\big[x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1}\,|\,\mathcal{F}_t\big]\Big] &= -\frac{1}{N}x_N^T P(\hat{\vartheta}_N)x_N\\
+\frac{1}{N}x_0^T P(\hat{\vartheta}_0)x_0 + \frac{1}{N}\sum_{t=0}^{N-1}&\Big[x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1} - E\big[x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1}\,|\,\mathcal{F}_t\big]\Big].\end{aligned}$$

The term $\frac{1}{N}x_0^T P(\hat{\vartheta}_0)x_0$ equals zero. As for $\frac{1}{N}x_N^T P(\hat{\vartheta}_N)x_N$, observe that

$$\frac{1}{N}x_N^T P(\hat{\vartheta}_N)x_N \le k_1\frac{1}{N}\|x_N\|^2,$$

k_1 being a suitable constant, since $P(\vartheta)$ is uniformly bounded on the compact set Θ (see Remark 2.3). Therefore, from (4.9) we get

$$\lim_{N\to\infty}\frac{1}{N}x_N^T P(\hat{\vartheta}_N)x_N = 0.$$

Define $w_t := x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1} - E\big[x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1}\,|\,\mathcal{F}_t\big]$. Then $\{w_t\}$ is a martingale difference. Hence, $\frac{1}{N}\sum_{t=0}^{N-1} w_t$ asymptotically vanishes if $\sum_{t=0}^{\infty}\frac{1}{t^2}E[w_{t+1}^2|\mathcal{F}_t] < \infty$ (see Theorem 2.18 in [16]). We have

$$E[w_{t+1}^2|\mathcal{F}_t] \le E\big[(x_{t+1}^T P(\hat{\vartheta}_{t+1})x_{t+1})^2|\,\mathcal{F}_t\big] \le k_2 E\big[\|x_{t+1}\|^4|\,\mathcal{F}_t\big] \le k_3\Big[\|x_t\|^4 + \|u_t\|^4 + 1\Big],$$

k_2, k_3 being suitable constants, since $P(\vartheta)$ is uniformly bounded over Θ and $\{n_t\}$ satisfies point 1 in Assumption 2.1. We then need to prove that $\sum_{t=0}^{\infty} \frac{1}{t^2}[\|x_t\|^4 + \|u_t\|^4] < \infty$. This is easily shown through (4.9) with $p = 8$, which implies $\|x_t\|^4 = o(t^{1/2})$ and $u_t^4 = o(t^{1/2})$, since $\sum_{t=0}^{\infty} \frac{1}{t^2}[\|x_t\|^4 + \|u_t\|^4] = \sum_{t=0}^{\infty} \frac{1}{t^{3/2}} \frac{1}{t^{1/2}}[\|x_t\|^4 + \|u_t\|^4]$, where $\sum_{t=0}^{\infty} \frac{1}{t^{3/2}}$ converges.

Second term:

Observe that $\{v_t\} := \{x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1} - E\big[x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1} \,|\, \mathcal{F}_t\big]\}$ is a martingale difference. By derivations similar to those for the first term, we can prove that $\frac{1}{N}\sum_{t=0}^{N} v_t \to 0$. Then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} E\big[x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1} \,|\, \mathcal{F}_t\big] = 0$$

is proven by showing that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1} = 0. \tag{4.13}$$

To prove (4.13), apply the Schwarz inequality to obtain

$$\begin{aligned}\left|\frac{1}{N}\sum_{t=0}^{N-1} x_{t+1}^T(P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1}))x_{t+1}\right| &\le \frac{1}{N}\sum_{t=0}^{N-1} \|P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1})\| \, \|x_{t+1}\|^2 \\ &\le \left(\frac{1}{N}\sum_{t=0}^{N-1} \|P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1})\|^2\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{t=0}^{N-1} \|x_{t+1}\|^4\right)^{\frac{1}{2}}.\end{aligned}$$

By Theorem 4.1 $\frac{1}{N}\sum_{t=0}^{N-1} \|x_{t+1}\|^4$ is bounded. Moreover, $\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} \|P(\hat{\vartheta}_t) - P(\hat{\vartheta}_{t+1})\|^2 = 0$ because of property (iii) in Theorem 3.2 and the Lipschitz continuity of $P(\vartheta)$ over Θ ($P(\vartheta)$ is analytic on Θ and Θ is compact). This concludes the proof of (4.13).

Third term:

Since $\hat{\vartheta}_t \in \Theta$ and $P(\vartheta)$ is uniformly bounded on Θ, then

$$0 \le \frac{1}{N}\sum_{t=0}^{N-1} E\big[e_t^T C^T P(\hat{\vartheta}_t) C e_t \,|\, \mathcal{F}_t\big] = \frac{1}{N}\sum_{t=0}^{N-1} e_t^T C^T P(\hat{\vartheta}_t) C e_t \le h_1 \frac{1}{N}\sum_{t=0}^{N-1} e_t^2,$$

h_1 being a suitable constant. We now show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} e_t^2 = 0 \quad \text{a.s.} \tag{4.14}$$

From Proposition 3.3 it follows that there exists a set of instant points $\mathcal{B}_{N-1}$ whose cardinality is upper bounded by a constant $C_{\mathcal{B}} < \infty$ such that $\frac{1}{N}\sum_{t=0,\ t\notin\mathcal{B}_{N-1}}^{N-1} e_t^2 = \frac{1}{N} o(\sum_{t=0}^{N-1} \|\varphi_t\|^2)$ a.s. Then, recalling the definition (3.6) of e_t, we have

$$\frac{1}{N}\sum_{t=0}^{N-1} e_t^2 = \frac{1}{N} o\left(\sum_{t=0}^{N-1} \|\varphi_t\|^2\right) + \frac{1}{N}\sum_{t\in\mathcal{B}_{N-1}} |\varphi_t^T(\vartheta^\circ - \hat{\vartheta}_t)|^2.$$

By Theorem 4.1, the first term tends to zero. As for the second term, we have that it can be upper bounded as follows:

$$\frac{1}{N}\sum_{t\in\mathcal{B}_{N-1}}|\varphi_t^T(\vartheta^\circ-\hat{\vartheta}_t)|^2\leq h_2\,C_{\mathcal{B}}\frac{1}{N}\max_{0\leq t\leq N-1}\|\varphi_t\|^2,\ \ h_2=\text{suitable constant,}$$

since $\hat{\vartheta}_t$ is bounded uniformly in time. Noting that $\|\varphi_t\|^2\leq\|x_{t+1}\|^2+\|x_t\|^2$, from (4.9) we get

$$\|\varphi_t\|^2=o(t)\quad\text{a.s.,}\tag{4.15}$$

which implies $\frac{1}{N}\max_{0\leq t\leq N-1}\|\varphi_t\|^2\to 0$.

Fourth term:

$$\begin{aligned}\frac{1}{N}\sum_{t=0}^{N-1}E\big[x_{t+1}^TP(\hat{\vartheta}_t)Ce_t\,|\,\mathcal{F}_t\big]&\\&=\frac{1}{N}\sum_{t=0}^{N-1}E\big[(A(\vartheta^\circ)x_t+B(\vartheta^\circ)u_t+Cn_{t+1})^TP(\hat{\vartheta}_t)Ce_t\,|\,\mathcal{F}_t\big]\\&=\frac{1}{N}\sum_{t=0}^{N-1}x_t^TA(\vartheta^\circ)^TP(\hat{\vartheta}_t)Ce_t+\frac{1}{N}\sum_{t=0}^{N-1}u_t^TB(\vartheta^\circ)^TP(\hat{\vartheta}_t)Ce_t.\end{aligned}$$

We next show that each term on the right-hand side goes to zero as N tends to infinity.

Since $\hat{\vartheta}_t\in\Theta$, with Θ compact, and $P(\vartheta)$ is analytic on Θ, by using Schwarz inequality, we have

$$\left|\frac{1}{N}\sum_{t=0}^{N-1}x_t^TA(\vartheta^\circ)^TP(\hat{\vartheta}_t)Ce_t\right|\leq k\left(\frac{1}{N}\sum_{t=0}^{N-1}\|x_t\|^2\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{t=0}^{N-1}e_t^2\right)^{\frac{1}{2}}.$$

Then $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}x_t^TA(\vartheta^\circ)^TP(\hat{\vartheta}_t)Ce_t=0$ a.s. follows from Theorem 4.1 and (4.14).

Similarly, one can prove that $\lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}u_t^TB(\vartheta^\circ)^TP(\hat{\vartheta}_t)Ce_t=0$ a.s. □

5. Conclusions. The more commonly adopted strategy for the design of adaptive control laws is the certainty equivalence approach. Although the approach is conceptually simple, working out stability and optimality results for certainty equivalence adaptive control schemes is a difficult task even in the ideal case when the true system belongs to the model class. This is due to the intricate interaction between control and identification in closed-loop, which can cause identifiability problems.

We introduced a new LQG adaptive control scheme based on the certainty equivalence principle able to ensure both stability and optimality results irrespectively of the excitation characteristics of the involved signals by adopting a cost-biased approach.

This paper presents the following limitations:

- *the true system is described as an ARX system subject to white noise.* This hypothesis is necessary mainly for the applicability of the proposed cost-biased LS identification method, whose properties are in fact derived on the basis of the LS estimate properties. As a consequence of this fact, the extension to the ARMAX system case is not straightforward.

• *the proposed identification method is nonrecursive.* The cost-biased identification index has, in general, multiple local minima and its minimization is not straightforward. Therefore, it should be minimized by resorting to some global optimization algorithm (see, e.g., [30, 31, 4, 15]). This limitation must be removed by introducing some recursive way to minimize our performance index so as to retain all the properties relevant to control.

These problems constitute interesting research issues. In particular, inspired by the result obtained for the white noise case, one can conceive of introducing appropriate cost-biased identification algorithms for the colored noise case. In this regard, much work has to be done, but an encouraging starting point is represented by the fact that the extended LS algorithm satisfies closed-loop properties similar to those valid for the LS algorithm (see, e.g., [10]).

Appendix. Proofs of the results in section 3.

Proof of Proposition 3.1. Recall that $\hat{\vartheta}_t \in \Theta$, $t \geq 0$, where Θ is compact and is such that all the parametrizations in Θ correspond to stabilizable models. We start by proving that $T(\vartheta) := \inf\{\tau \in Z_+ : \|F(\vartheta)^\tau\| \leq \mu\}$ is uniformly bounded in the compact set Θ, i.e., $\sup_{\vartheta \in \Theta} T(\vartheta) < \infty$.

Condition $\vartheta \in \Theta$ implies that the system $\mathcal{A}(\vartheta; q^{-1}) y_{t+1} = \mathcal{B}(\vartheta; q^{-1}) u_t$ associated with parameter ϑ is stabilizable and therefore stabilized by the control law $u_t = \mathcal{S}(\vartheta; q^{-1}) y_t + \mathcal{R}(\vartheta; q^{-1}) u_t$. From this it follows that the dynamic matrix $F(\vartheta)$ of the time-invariant system

$$x_{t+1} = F(\vartheta) x_t \tag{A.1}$$

is exponentially stable.

Denote by $\{\lambda_i(\vartheta)\}_{i=1,\ldots,n+m-1}$ the eigenvalues of $F(\vartheta)$.

By the observation that $F(\vartheta)$ is a continuous function of ϑ, $\mathcal{C} = \{\vartheta \in \Re^{n+m} : q^s \mathcal{A}(\vartheta; q^{-1})$ and $q^{s-1} \mathcal{B}(\vartheta; q^{-1})$ have no unstable pole-zero cancellations$\}$, we have that $\bar{\lambda}(\vartheta) := \max_{i \in \{1,\ldots,n+m-1\}} |\lambda_i(\vartheta)|$ is also a continuous function of ϑ, $\vartheta \in \mathcal{C}$. Being Θ compact and included in $\mathcal{C}$, the conclusion is finally drawn that

$$\bar{\lambda} := \max_{\vartheta \in \Theta} \bar{\lambda}(\vartheta) < 1.$$

Fix now a real number $\bar{\nu} \in (\bar{\lambda}, 1)$ and introduce the system

$$w_{t+1} = \frac{1}{\bar{\nu}} F(\vartheta) \, w_t. \tag{A.2}$$

System (A.2) is exponentially stable $\forall\, \vartheta \in \Theta$, since $|\frac{\lambda_i(\vartheta)}{\bar{\nu}}| \leq \frac{\bar{\lambda}}{\bar{\nu}} < 1 \;\forall\, i, \;\forall\, \vartheta \in \Theta$. Hence, the solution $S(\vartheta)$ to the Lyapunov equation associated with matrix $\frac{1}{\bar{\nu}} F(\vartheta)$

$$\frac{1}{\bar{\nu}} F(\vartheta)^T S(\vartheta) \frac{1}{\bar{\nu}} F(\vartheta) - S(\vartheta) = -I$$

is positive definite. Moreover, it is a standard fact that the state vector w_t of system (A.2) can be bounded as follows in terms of $S(\vartheta)$:

$$\|w_t\| \leq \sqrt{\frac{\lambda_{max}(S(\vartheta))}{\lambda_{min}(S(\vartheta))}} \; \| \, w_{t^\star} \|, \;\; t \geq t^\star \geq 0, \tag{A.3}$$

where $\lambda_{max}(S(\vartheta))$ and $\lambda_{min}(S(\vartheta))$ are, respectively, the maximum and minimum eigenvalues of $S(\vartheta)$. Since $S(\vartheta)$ is continuous in the closed set Θ (see [12]), we can define $c := \max_{\vartheta\in\Theta} \sqrt{\frac{\lambda_{max}(S(\vartheta))}{\lambda_{min}(S(\vartheta))}}$ and rewrite inequality (A.3) as $\|w_t\| \le c\,\|w_{t^\star}\|,\ \ t \ge\ t^\star$ $\forall\,\vartheta \in \Theta$. Setting $w_{t^\star} = x_{t^\star}$, we finally get a bound on the state vector x_t of the time-invariant system (A.1)

$$\|x_t\| \le c\,\bar\nu^{t-t^\star}\|x_{t^\star}\|, \quad t \ge t^\star, \forall\,\vartheta \in \Theta. \tag{A.4}$$

Set $\bar T = \inf\{\tau \in Z_+ : c\,\bar\nu^\tau \le \mu\} < \infty$. Since $\|x_{\bar T + t^\star}\| = \|F(\vartheta)^{\bar T}\,x_{t^\star}\| \le \mu\|x_{t^\star}\|$ $\forall\,\vartheta \in \Theta$, $\forall\,x_{t^\star}$, then $\|F(\vartheta)^{\bar T}\| = \sup_{\|x\|\ne 0} \frac{\|F(\vartheta)^{\bar T}\,x\|}{\|x\|} \le \mu\ \forall\,\vartheta \in \Theta$, and therefore $T(\vartheta) = \inf\{\tau \in Z_+ : \|F(\vartheta)^\tau\| \le \mu\}$ satisfies $T(\vartheta) \le \bar T\ \forall\,\vartheta \in \Theta$. This finally implies that

$$\sup_{\vartheta\in\Theta}\{T(\vartheta)\} \le \bar T < \infty. \tag{A.5}$$

Let us turn now to considering the time-varying system $x_{t+1} = F(\hat\vartheta_t)\,x_t$.

Being $\hat\vartheta_t \in \Theta$, $t \ge 0$, from (A.5) it follows that the updating time interval T_i in (3.4) is uniformly bounded:

$$T := \sup_{i\ge 0} T_i < \bar T. \tag{A.6}$$

We are now in a position to establish the uniform exponential stability. We apply (A.4) to the state vector x_t on each finite time interval $[t_i, t_{i+1}]$, thus getting

$$\|x_t\| \le c\,\bar\nu^{t-t^\star}\|x_{t^\star}\|, \ \ t_i \le t^\star \le t \le t_{i+1}. \tag{A.7}$$

If we choose $t^\star = t_i$, we have $\|x_t\| \le c\,\bar\nu^{t-t_i}\|x_{t_i}\|,\ \ t \in [t_i, t_{i+1}]$. From the definition (3.4) of $\{T_k\}$, it follows that $\|x_{t_i}\| \le \mu^{i-j}\|x_{t_j}\|,\ \ j \le i$. By applying (A.7) in the time interval $[t_{j-1}, t_j]$ with $t = t_j$, we get $\|x_{t_j}\| \le c\,\bar\nu^{t_j - t^\star}\|x_{t^\star}\|,\ \ t^\star \in [t_{j-1}, t_j]$. These last three inequalities lead to

$$\|x_t\| \le c\,\bar\nu^{t-t_i}\mu^{i-j}\,c\,\bar\nu^{t_j-t^\star}\|x_{t^\star}\|, \ \ t_{j-1} \le t^\star \le t_j \le t_i \le t \le t_{i+1}, \ \ j \le i.$$

By setting $\nu = \max\{\bar\nu, \mu^{\frac{1}{T}}\} < 1$, we have that $\mu \le \nu^{T_k}\ \forall k$ and therefore

$$\begin{aligned}\|x_t\| &\le c^2\,\nu^{t-t_i}\nu^{t_i-t_{i-1}}\dots\nu^{t_{j+1}-t_j}\nu^{t_j-t^\star}\|x_{t^\star}\| \\ &= c^2\,\nu^{t-t^\star}\|x_{t^\star}\|, \ \ t_{j-1} \le t^\star \le t_j \le t_i \le t \le t_{i+1}.\end{aligned}$$

Finally, from this last inequality and inequality (A.7), we get $\|x_t\| \le c^2\,\nu^{t-t^\star}\|x_{t^\star}\|$, $t^\star \le t$, i.e., the thesis.

Proof of Theorem 3.2. Point (i): $D_t(\vartheta) - V_t(\hat\vartheta_t^{LS})$ can be written as follows:

$$\begin{aligned} D_t(\vartheta) - V_t(\hat\vartheta_t^{LS}) &= \sum_{s=1}^t (y_s - \varphi_{s-1}^T\vartheta)^2 + \alpha_t J^\star(\vartheta) + \gamma_t\|\vartheta - \hat\vartheta_{t-1}\| - \sum_{s=1}^t (y_s - \varphi_{s-1}^T\hat\vartheta_t^{LS})^2 \\ &= \vartheta^T\sum_{s=1}^t \varphi_{s-1}\varphi_{s-1}^T\vartheta - 2\vartheta^T\sum_{s=1}^t\varphi_{s-1}y_s + \alpha_t J^\star(\vartheta) + \gamma_t\|\vartheta - \hat\vartheta_{t-1}\| \qquad \text{(A.8)} \\ &\quad -(\hat\vartheta_t^{LS})^T\sum_{s=1}^t\varphi_{s-1}\varphi_{s-1}^T\hat\vartheta_t^{LS} + 2(\hat\vartheta_t^{LS})^T\sum_{s=1}^t\varphi_{s-1}y_s. \end{aligned}$$

The LS estimate $\hat{\vartheta}_t^{LS}$ minimizing $V_t(\vartheta)$ satisfies the following equality:

$$\sum_{s=1}^{t} \varphi_{s-1} y_s = \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T \hat{\vartheta}_t^{LS}.$$

Substituting this last expression in (A.8), we obtain

$$\begin{aligned} D_t(\vartheta) - V_t(\hat{\vartheta}_t^{LS}) = (\vartheta - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta - \hat{\vartheta}_t^{LS}) \\ + \alpha_t J^\star(\vartheta) + \gamma_t \|\vartheta - \hat{\vartheta}_{t-1}\|. \end{aligned} \tag{A.9}$$

Set $\vartheta_t := \arg\min_{\vartheta\in\Theta} D_t(\vartheta)$. By definition of ϑ_t we have

$$D_t(\vartheta_t) - V_t(\hat{\vartheta}_t^{LS}) \leq D_t(\vartheta) - V_t(\hat{\vartheta}_t^{LS}), \quad \vartheta \in \Theta.$$

By choosing $\vartheta = \vartheta^\circ$ and using expression (A.9), we then get

$$\begin{aligned} &(\vartheta_t - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta_t - \hat{\vartheta}_t^{LS}) + \alpha_t J^\star(\vartheta_t) + \gamma_t \|\vartheta_t - \hat{\vartheta}_{t-1}\| \\ &\leq (\vartheta^\circ - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta^\circ - \hat{\vartheta}_t^{LS}) + \alpha_t J^\star(\vartheta^\circ) + \gamma_t \|\vartheta^\circ - \hat{\vartheta}_{t-1}\| \\ &= O(\alpha_t) \quad \text{a.s.}, \end{aligned} \tag{A.10}$$

where the last equality follows from Theorem 2.4, the definition (3.5) of α_t, the fact that $\|\vartheta^\circ - \hat{\vartheta}_{t-1}\|$ is bounded, and the relation $\gamma_t = o(\alpha_t)$. Since $\alpha_t J^\star(\vartheta_t) + \gamma_t\|\vartheta_t - \hat{\vartheta}_{t-1}\| \geq 0$, we have $(\vartheta_t - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta_t - \hat{\vartheta}_t^{LS}) = O(\alpha_t)$ a.s. From definition (3.5) of α_t and Theorem 2.4, we then have

$$\begin{aligned} (\vartheta_t - \vartheta^\circ)^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta_t - \vartheta^\circ) &\leq 2\Big[(\vartheta_t - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta_t - \hat{\vartheta}_t^{LS}) \\ &\quad + (\hat{\vartheta}_t^{LS} - \vartheta^\circ)^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\hat{\vartheta}_t^{LS} - \vartheta^\circ)\Big] \\ &= O(\alpha_t) \text{ a.s.}, \end{aligned}$$

thus concluding the proof of point (i), since $\hat{\vartheta}_t = \vartheta_t$, for $t = t_i$, $i = 0, 1, \ldots$.

Point (ii): A simple elaboration of (A.10) shows that

$$\begin{aligned} J^\star(\vartheta_t) &\leq \frac{(\vartheta^\circ - \hat{\vartheta}_t^{LS})^T \sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T (\vartheta^\circ - \hat{\vartheta}_t^{LS})}{\alpha_t} + J^\star(\vartheta^\circ) + \frac{\gamma_t}{\alpha_t}\|\vartheta^\circ - \hat{\vartheta}_{t-1}\| \\ &= \frac{O(\log \lambda_{max}(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T))}{\log^{1+\delta} \lambda_{max}(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T)} + \frac{o(\alpha_t)}{\alpha_t} + J^\star(\vartheta^\circ) \text{ a.s.}, \end{aligned}$$

where in the second equation we have used the definition of α_t given in equation (3.5) and the fact that $\gamma_t = o(\alpha_t)$. To conclude the proof, it suffices to show that $\lim_{t\to\infty} \log \lambda_{max}(\sum_{s=1}^{t} \varphi_{s-1}\varphi_{s-1}^T) = \infty$. The easy proof of this fact is omitted.

Point (iii): By the definition (2.8) of $\hat{\vartheta}_t$, we have

$$V_t(\hat{\vartheta}_t) + \alpha_t J^\star(\hat{\vartheta}_t) + \gamma_t\|\hat{\vartheta}_t - \hat{\vartheta}_{t-1}\| \leq V_t(\hat{\vartheta}_{t-1}) + \alpha_t J^\star(\hat{\vartheta}_{t-1}),$$

which implies

$$\text{(A.11)}\qquad \sum_{t=1}^{N}\gamma_t\|\hat{\vartheta}_t-\hat{\vartheta}_{t-1}\| \le \sum_{t=1}^{N}[V_t(\hat{\vartheta}_{t-1})-V_t(\hat{\vartheta}_t)] + \sum_{t=1}^{N}\alpha_t[J^\star(\hat{\vartheta}_{t-1})-J^\star(\hat{\vartheta}_t)].$$

The first term in the right-hand side of (A.11) can be bounded as follows:

$$\begin{aligned}\sum_{t=1}^{N}[V_t(\hat{\vartheta}_{t-1})-V_t(\hat{\vartheta}_t)] &\le V_1(\hat{\vartheta}_0)-V_N(\hat{\vartheta}_N)+\sum_{t=1}^{N-1}[V_{t+1}(\hat{\vartheta}_t)-V_t(\hat{\vartheta}_t)]\\ &\le V_1(\hat{\vartheta}_0)+\sum_{t=1}^{N-1}[\varphi_t^T(\vartheta^\circ-\hat{\vartheta}_t)+n_{t+1}]^2\\ &\le V_1(\hat{\vartheta}_0)+2\sum_{t=1}^{N-1}[\varphi_t^T(\vartheta^\circ-\hat{\vartheta}_t)]^2+2\sum_{t=1}^{N-1}n_{t+1}^2\\ &\le k_1\left[1+\sum_{t=1}^{N}\|\varphi_{t-1}\|^2+\sum_{t=1}^{N-1}n_{t+1}^2\right],\end{aligned}$$

k_1 being a suitable constant, where we used the boundedness of $\hat{\vartheta}_t$.

By Remark 2.3, the second term in the right-hand side of (A.11) can be bounded as follows:

$$\begin{aligned}\sum_{t=1}^{N}\alpha_t[J^\star(\hat{\vartheta}_{t-1})-J^\star(\hat{\vartheta}_t)] &= \alpha_1 J^\star(\hat{\vartheta}_0)-\alpha_N J^\star(\hat{\vartheta}_N)+\sum_{t=1}^{N-1}(\alpha_{t+1}-\alpha_t)J^\star(\hat{\vartheta}_t)\\ &\le \alpha_1 J^\star(\hat{\vartheta}_0)+\max_{\vartheta\in\Theta}J^\star(\vartheta)\sum_{t=1}^{N-1}(\alpha_{t+1}-\alpha_t)\\ &= k_2[1+\alpha_N],\end{aligned}$$

where k_2 is a suitable constant.

Substituting these bounds in (A.11), we then have

$$\text{(A.12)}\qquad \frac{1}{N}\sum_{t=1}^{N}\gamma_t\|\hat{\vartheta}_t-\hat{\vartheta}_{t-1}\| \le \bar{k}\left[\frac{1}{N}+\frac{\alpha_N}{N}+\frac{1}{N}\sum_{t=1}^{N}\|\varphi_{t-1}\|^2+\frac{1}{N}\sum_{t=1}^{N-1}n_{t+1}^2\right],$$

with $\bar{k}$ = suitable constant. Observe now that all the terms in the right-hand side of (A.12) are $O(1)$. This, in particular, follows from the assumption of point (iii) in Theorem 3.2 that $\sum_{t=1}^{N}\|\varphi_{t-1}\|^2 = O(N)$ and Assumption 2.1, point 2. Then $\frac{1}{N}\sum_{t=1}^{N}\gamma_t\|\hat{\vartheta}_t-\hat{\vartheta}_{t-1}\| = O(1)$. Since γ_t tends to infinity, this last equation implies $\frac{1}{N}\sum_{t=1}^{N}\|\hat{\vartheta}_t-\hat{\vartheta}_{t-1}\| = o(1)$, that is, the thesis.

Proof of Proposition 3.3. Fix a real number $\epsilon > 0$ and a time instant N. Consider the set of instant points in the interval $[0, N]$ where $\tilde{\vartheta}_t := \vartheta^\circ - \hat{\vartheta}_t$ changes: $t_0, t_1, \dots, t_{i(N)}$, where $i(N) := \max\{i : t_i \le N\}$. In these instant points we define a set of subspaces $\{S_{t_i}\}_{i=0}^{i(N)}$ through the following backward recursive procedure:

for $i = i(N) + 1$, set $S_i = \emptyset$,

for $i = i(N), i(N) - 1, \dots, 0$, set (here and throughout the symbol $\tilde{\vartheta}_{t,S}$ stands for the projection of vector $\tilde{\vartheta}_t$ onto the subspace S)

$$\text{(A.13)}\qquad S_{t_i} = \begin{cases} S_{t_{i+1}} & \text{if } \|\tilde{\vartheta}_{t_i,S^{\perp}_{t_{i+1}}}\| \le \epsilon,\\ S_{t_{i+1}} \oplus span\{\tilde{\vartheta}_{t_i}\} & \text{otherwise.}\end{cases}$$

For each $t \in [0, N]$, with the notation $i(t) := \max\{i : t_i \leq t\}$, we have

$$|\varphi_t^T \tilde{\vartheta}_t|^p \leq c_p \, |\varphi^T_{t,S^\perp_{t_{i(t)}}} \tilde{\vartheta}_{t,S^\perp_{t_{i(t)}}}|^p + c_1 \, |\varphi^T_{t,S_{t_{i(t)}}} \tilde{\vartheta}_{t,S_{t_{i(t)}}}|^p, \tag{A.14}$$

where c_p is a suitable constant depending on p. By definition (A.13), the first term in the right-hand side can be upper bounded as follows:

$$|\varphi^T_{t,S^\perp_{t_{i(t)}}} \tilde{\vartheta}_{t,S^\perp_{t_{i(t)}}}|^p \leq \epsilon^p \|\varphi_t\|^p. \tag{A.15}$$

To handle the second term, we first work out a basis in $S_{t_{i(t)}}$. For this purpose, consider the subset $\{\tau_j\}_{j=1}^{dim(S_{t_0})}$ of instant points $\{t_i\}_{i=0}^{i(N)}$ such that subspace S_{t_i} enlarges: $S_{\tau_j} \supset S_{t_i}$, $t_i > \tau_j$. The searched basis is $\{\tilde{\vartheta}_{\tau_j}\}_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})}$.

In view of the uniform boundedness of $\tilde{\vartheta}_t$ and also considering the very definition of subspaces S_{t_i} (equation (A.13)), it is easy to see that vectors $\{\tilde{\vartheta}_{\tau_j}\}$ are spread in subspace $S_{t_{i(t)}}$ in such a way that the angle between each pair of vectors tends to zero only when $\epsilon \to 0$. Consequently, there exists a constant $c(\epsilon)$, depending on ϵ, but independent of N, such that term $|\varphi^T_{t,S_{t_{i(t)}}} \tilde{\vartheta}_{t,S_{t_{i(t)}}}|^p$ in the right-hand side of inequality (A.14) can be bounded as follows:

$$\begin{aligned} |\varphi^T_{t,S_{t_{i(t)}}} \tilde{\vartheta}_{t,S_{t_{i(t)}}}|^p &\leq \Delta^p \|\varphi_{t,S_{t_{i(t)}}}\|^p \\ &\leq \Delta^p c(\epsilon) \sum_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p, \end{aligned} \tag{A.16}$$

where $\Delta = \max_{\vartheta_1,\vartheta_2 \in \Theta} \|\vartheta_1 - \vartheta_2\|$.

By plugging estimates (A.15) and (A.16) in (A.14), we obtain

$$|\varphi_t^T \tilde{\vartheta}_t|^p \leq c_p \epsilon^p \|\varphi_t\|^p + c_p \, \Delta^p \, c(\epsilon) \sum_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p.$$

Summing up these relations from time $t = 0$ to $t = N$, we finally have

$$\sum_{t=0}^{N} |\varphi_t^T \tilde{\vartheta}_t|^p \leq c_p \epsilon^p \sum_{t=0}^{N} \|\varphi_t\|^p + c_p \, \Delta^p \, c(\epsilon) \sum_{t=0}^{N} \sum_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p. \tag{A.17}$$

Introduce now the time-varying set of instant points

$$\mathcal{B}_N := \cup_{j=1}^{dim(S_{t_0})} \{\tau_j, \tau_j + 1, \ldots, \tau_j + T - 1\},$$

where $T := \sup_{i \geq 0} T_i < \infty$ (see (A.6) in the proof of Proposition 3.1). Since $\dim(S_{t_0}) \leq n + m$, we obviously have $|\mathcal{B}_N| \leq T(n + m)$.

Then

$$\sum_{t=0,\, t \notin \mathcal{B}_N}^{N} \sum_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p \leq \sum_{j=1}^{dim(S_{t_0})} \sum_{t=0}^{\tau_j - 1} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p.$$

We now show that

$$\sum_{t=0}^{t_i-1} |\varphi_t^T \tilde{\vartheta}_{t_i}|^p = o\left(\sum_{t=0}^{t_i-1} \|\varphi_t\|^p\right) \quad \text{a.s.,} \tag{A.18}$$

from which it follows that

$$\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} \sum_{j=dim(S_{t_0})-dim(S_{t_{i(t)}})+1}^{dim(S_{t_0})} \|\varphi_{t,span\{\tilde{\vartheta}_{\tau_j}\}}\|^p \leq \frac{n+m}{\epsilon^p}\left[o\left(\sum_{t=0}^{N}\|\varphi_t\|^p\right) + O(1)\right], \tag{A.19}$$

where we used the fact that $\dim(S_{t_0}) \leq n+m \ \forall N$.

Observe first that

$$\sum_{t=0}^{t_i-1} \|\varphi_t\|^2 = O\left(\sum_{t=0}^{t_i-1} \|\varphi_t\|^p\right) \quad \text{a.s.} \tag{A.20}$$

Indeed, using Jensen's inequality [11, Corollary 1 in section 4.3])

$$\begin{aligned}\sum_{t=0}^{t_i-1} \|\varphi_t\|^2 &= t_i\left[\left(\frac{1}{t_i}\sum_{t=0}^{t_i-1}\|\varphi_t\|^2\right)^{p/2}\right]^{2/p} \\ &\leq t_i\left[\frac{1}{t_i}\sum_{t=0}^{t_i-1}\|\varphi_t\|^p\right]^{2/p} = \sum_{t=0}^{t_i-1}\|\varphi_t\|^p\left[\frac{t_i}{\sum_{t=0}^{t_i-1}\|\varphi_t\|^p}\right]^{1-2/p},\end{aligned}$$

where

$$\limsup_{i\to\infty} \frac{t_i}{\sum_{t=0}^{t_i-1}\|\varphi_t\|^p} < \infty \quad \text{a.s.} \tag{A.21}$$

This last equation is easily derived as follows. From the regression-like form $y_t = \varphi_{t-1}^T\vartheta^\circ + n_t$, it follows that $|n_t|^p \leq 2^{p-1}\max\{\|\vartheta^\circ\|, 1\}[\,|y_t|^p + \|\varphi_{t-1}\|^p\,]$. Taking into account that the autoregressive part of system is not trivial ($n > 0$), this in turn implies that $|n_t|^p \leq h_1[\,\|\varphi_t\|^p + \|\varphi_{t-1}\|^p\,]$, from which it is easily shown that $\sum_{t=1}^{N-1}|n_t|^p \leq h_1\sum_{t=0}^{N-1}\|\varphi_t\|^p$, where h_1 is a suitable constant. Since $\frac{1}{N}\sum_{t=1}^{N-1}|n_t|^p \geq [\frac{1}{N}\sum_{t=0}^{N-1}n_t^2]^{p/2}$ (using Jensen's inequality), from Assumption 2.1, we then get

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^p > 0 \quad \text{a.s.,}$$

from which (A.21) follows.

By means of (A.20), we now show that $\sum_{t=0}^{t_i-1}|\varphi_t^T\tilde{\vartheta}_{t_i}|^p = o(\sum_{t=0}^{t_i-1}\|\varphi_t\|^p)$ a.s., which implies (A.18). This equation is easily derived from property (i) in Theorem 3.2

as follows:

$$\begin{aligned}\sum_{t=0}^{t_i-1} |\varphi_t^T \tilde{\vartheta}_{t_i}|^p &\le \left|\sum_{t=0}^{t_i-1} |\varphi_t^T(\vartheta^\circ - \vartheta_{t_i})|^2\right|^{p/2} \\ &= o\left(\left(\operatorname{Log} \sum_{t=0}^{t_i-1} \|\varphi_t\|\right)^{p(1+\delta)/2}\right) \qquad \text{(by property (i))} \\ &= o\left(\sum_{t=0}^{t_i-1} \|\varphi_t\|^2\right) \\ &= o\left(\sum_{t=0}^{t_i-1} \|\varphi_t\|^p\right) \qquad \text{(by (A.20))}.\end{aligned}$$

By using inequality (A.17) and inequality (A.19), we obtain

$$\begin{aligned}\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} |\varphi_t^T \tilde{\vartheta}_t|^p &\le c_p\epsilon^p \sum_{t=0}^{N} \|\varphi_t\|^p + c_p \Delta^p c(\epsilon)\frac{n+m}{\epsilon^p}\left[o\left(\sum_{t=0}^{N} \|\varphi_t\|^p\right) + O(1)\right] \\ &\le c_p\epsilon^p O\left(\sum_{t=0}^{N} \|\varphi_t\|^p + N\right) + c_p \Delta^p c(\epsilon)\frac{n+m}{\epsilon^p} o\left(\sum_{t=0}^{N} \|\varphi_t\|^p + N\right),\end{aligned}$$

which finally implies that

$$\limsup_{N\to\infty} \frac{\displaystyle\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} |\varphi_t^T \tilde{\vartheta}_t|^p}{\displaystyle\sum_{t=0}^{N} \|\varphi_t\|^p + N} \le c_p \epsilon^p.$$

Since ϵ is arbitrarily chosen, the thesis follows.

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