IDENTIFICATION WITH FINITELY MANY DATA POINTS: THE LSCR APPROACH

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PART I: From nominal-model to model-set identification
true

\[ \mathcal{P} \]

\[ \sim \]

\[ \hat{\mathcal{P}} \]

identified
\[ \Pr\{\theta^o = \hat{\theta}\} = 0 \]
\[ \Pr\{\theta^0 \in \text{region}\} \geq 0 \]
Example

\[ y_t = a^o y_{t-1} + b^o u_{t-1} + n_t \]
Goal: finding confidence regions, guaranteed under general assumptions
PART II:

$LSCR = \text{Leave-out Sign-dominant Correlation Regions}$
A simple Example

\[ y_t + a^0 y_{t-1} = w_t \]

\( w_t \) = independent, symmetrically distributed

Find a “guaranteed” interval for \( a^0 \)
The LSCR approach

\[ y_t + ay_{t-1} = w_t \]

\[ \hat{y}_t = -ay_{t-1} \]

\[ \epsilon_t(a) = y_t - \hat{y}_t = y_t + ay_{t-1} \]

\[
\frac{1}{N} \sum_{t=1}^{N} \epsilon_{t-1}(a) \epsilon_t(a)
\]

empirical correlation

this is a function of \(a\)
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<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
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\[ \Rightarrow \frac{1}{4}[\epsilon_1(a)\epsilon_2(a) + \epsilon_2(a)\epsilon_3(a) + \epsilon_4(a)\epsilon_5(a) + \epsilon_5(a)\epsilon_6(a)] \]

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\[ \ldots \]

\[ \ldots \]

\[ \ldots \]
\[ \frac{1}{4} \sum_{t=1,2,4,5} \epsilon_t(a^0) \epsilon_t(a^0) = \frac{1}{4} \sum_{t=1,2,4,5} w_{t-1} w_t \]
discard regions where empirical correlations are positive or negative “too many times” (LSCR = Leave-out Sign-dominant Correlation Regions)
Theorem (‘pivotal’ result)

For $a = a^0$, the value 0 is "at top", "as second top", ..., with the same probability $1/8$, independently of the noise characteristics.

$Pr\{a^0 \in interval\} = 0.5$
10 more trials
LSCR - general

$w_t$ independent of $u_t$
LSCR - general
LSCR - general

$H(\theta^0)$

$G(\theta^0)$

$u_t$ independent of $u_t$
LSCR - general

\[ H(\theta^o) \]

\[ G(\theta^o) \]

\[ w_t \] independent of \( u_t \)

\[ u_t, y_t, \ t = 1, ..., N \]

region guaranteed under general assumptions on noise
**LSCR - general**

- \( \epsilon_{t-d}(\theta) \epsilon_t(\theta) \)
- \( u_{t-d} \epsilon_t(\theta) \)

**Theorem**

\( \omega_t \) zero mean, symmetrically distributed
(no assumption on strength of noise)

\[
Pr \{ \theta^o \in \text{LSCR region} \} = 1 - \frac{q}{M+1}
\]

- usually, intersect more regions
Example 1

\[ y_t + a^0 y_{t-1} = w_t + c^0 w_{t-1} \]
\[ a^0 = -0.5, \quad c^0 = 0.2, \quad w_t \sim WGN(0, 1) \]

\[ N = 1025 \]
\[ \epsilon_{t-1}(a, c) \epsilon_t(a, c) \]
\[ \epsilon_{t-2}(a, c) \epsilon_t(a, c) \]
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Example 2

\[ F^0 = \frac{b^o z^{-1}}{1 + a^o z^{-1}} \quad a^o = -0.7, \quad b^o = 0.3 \]

\[ H^0 = 1 + h^o z^{-1} \quad h^o = 0.5 \]

\[ N = 2047 \]

\[ w_t \sim WGN(0, 1) \]

\[ u_t \sim WGN(0, 10^{-6}) \]
... use asymptotic theory
... use asymptotic theory

\[ F(\theta^o) \]

\[ F(\hat{\theta}) \]

\[ Pr = 90\% \]
... use LSCR

\[ Pr = 90\% \]
**LSCR - properties**

- The region shrinks around $\theta^o$
- For any $N$, $\theta^o \in$ region with given $Pr$ despite no assumption on level of noise is made
PART III: Extensions
• $w_t$ non-symmetrically distributed

• nonlinear systems (bi-linear systems)

UNMODELED DYNAMICS

• estimate $G(\theta^0)$ only

• unmodelled dynamics

\[ y_t = b_1^0 u_{t-1} + b_2^0 u_{t-2} + w_t \]
IN CONCLUSION:

LSCR is a new framework for sys id that provides guaranteed results with minimal assumptions and it represents an exciting topic of research.