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The Problem of Pole–Zero Cancellation in Transfer Function Identification and Application to Adaptive Stabilization*

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The controllability of the estimated model can be secured in a stochastic framework by a suitable modification of the least-squares algorithm, leading to an identification technique that can be safely used in any adaptive control scheme.

Key Words—Adaptive control; RLS identification; stochastic systems; controllability; pole–zero cancellation; adaptive stabilization.

Abstract—The asymptotic controllability of the identified system is a central problem in adaptive control. If controllability is ascertained, the analysis of even complex adaptive controllers based on multistep performance indices is drastically simplified. In this paper, we study the controllability issue in connection with the recursive least-squares (RLS) algorithm. We show that standard RLS does not generally provide models that are controllable. However, a variant of this method that preserves all the basic properties of the standard RLS and also guarantees asymptotic controllability is introduced. The algorithm can be safely used in any adaptive control system, provided that the control law is able to stabilize known invariant plants.

1. INTRODUCTION

Since the pioneering work of Åström and Wittenmark (1973), the area of self-tuning control has attracted an increasing amount of interest. In particular, over the last decade, the analysis of adaptive schemes has represented a stimulating challenge for control theorists, and much attention has been paid to the establishment of rigorous convergence results.

The first significant contribution in this direction is probably due to Goodwin *et al.* (1981). In this celebrated paper, the authors showed the self-optimality and mean-square stability of a minimum-variance adaptive tracker based on the stochastic approximation algorithm. The analysis in this paper was inherently based on the minimum-phase assumption of the system under control.

Ever since, much effort has been made in order to extend the results of Goodwin *et al.* (1981) to more general situations. In Sin and Goodwin (1982), a minimum-variance regulator based on a least-squares identification algorithm is considered. The self-optimality is achieved thanks to a suitable modification of the original least-squares technique. In analogy with the situation described in Goodwin *et al.* (1981), in the adaptive scheme of Sin and Goodwin (1982), the parameter consistency is not guaranteed, unless the reference signal is sufficiently rich. In order to get a strong consistent parameter estimate, Caines and Lafortune (1984) suggested the injection of an additive noise in the control action so as to improve the excitation characteristics of the signals. However, as noted in Chen and Guo (1987), this results in a degradation of the tracking accuracy. Asymptotic optimality can be recovered by letting the additive noise vanish in the long run (Chen and Guo, 1987). Both Caines and Lafortune (1984) and Chen and Guo (1987) are devoted to adaptive schemes based on the stochastic approximation algorithm. A full treatment of the self tuning regulator equipped with the extended-least squares algorithm can be found in the remarkable paper by Guo and Chen (1991).

A control law of minimum-variance type is a common characteristic of all the above quoted contributions. Since one of the peculiar features of such a technique is to cancel out the system zeros by the introduction of coincident poles in the regulator, the system is always required to be minimum-phase. Indeed, this constitutes a major limitation on the applicability of the theory developed in these contributions.

In a couple of interesting papers, Rootzen and

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Sternby (Sternby, 1977; Rootzen and Sternby, 1984) proved that the least-squares estimate generally converges even without any excitation condition. This result weakened the previously known convergence conditions to a great extent, and set the basis for a renewed attack on long-standing open problems in adaptive control. On the basis of the results of Sternby (1977) and Rootzen and Sternby (1984), Kumar (1990) analyzed the properties of a large class of adaptive regulators based on the recursive least-squares algorithm. By exploiting the normal equations, he was able to establish the stability and the optimality of the corresponding control scheme. Unfortunately, the general control law of Kumar is still governed by a dynamics that is factored as the product of two terms, the first being the numerator of the system transfer function. As a consequence, in analogy with the minimum-variance case, the minimum-phase assumption is still necessary.

Turning to the case of non-minimum-phase systems, the existing results are quite scarce. In this connection, note that the analyses of minimum-phase plants always rest—explicitly or implicitly—on the stability of the inverse system (which implies system input boundedness, given system output boundedness). Since this implication is obviously false for non-minimum-phase plants, the classical lines of reasoning developed for the minimum-phase case are not extendible, and new routes of analysis have to be discovered.

It is widely recognized (see e.g. Campi, 1994) that a drastic simplification in the analysis of general adaptive control laws applied to possibly non-minimum-phase plants is achieved if the following two conditions are ascertained:

(i) the parameters whose value actually influences the behavior of the controlled system are estimated with progressively increasing accuracy;

(ii) the asymptotically estimated model is controllable (i.e. it does not exhibit pole–zero cancellations).

Clearly, the fulfillment of the first requirement is necessary to enable the regulator to select a suitable control action for the system. However, this does not mean in general that the entire dynamics of the system must be learnt. In fact, only those parts that are excited by the input signals are responsible of its behavior, and therefore must be reliably estimated. As for the second condition, note that in adaptive control, the control action is computed based on the knowledge of the estimated model. Therefore lack of controllability of such a model leads to paralysis in the control selection.

Unfortunately, establishing the simultaneous satisfaction of conditions (i) and (ii) is not easy. This is why, in the literature, such conditions are often taken for granted (see e.g. Ren, 1993). Over the last decade, such a difficulty has represented one of the main stumbling blocks in adaptive control.

The goal of the present paper is to enlighten this matter in connection with the commonly used recursive least-squares (RLS) algorithm. We first review some facts concerning the convergence of this algorithm, and prove that the component of the parameter estimation error along the directions of diverging information is asymptotically vanishing (Section 2.2). This result is strictly related to condition (i) above. Unfortunately, RLS does not generally provide controllable models. This is proved through a simple counterexample in Section 2.3. A modified version of RLS is then introduced that guarantees asymptotic controllability. This is obtained by adding an extra term to the standard RLS estimate that has the fundamental feature that it tends to zero along the directions of diverging information where the system parameters are correctly estimated. In this way, an RLS-based algorithm that simultaneously meets conditions (i) and (ii) is obtained. The proposed algorithm can be successfully used in a variety of adaptive control problems. As a significant application, in Section 3, we derive a general result on adaptive stabilization that holds true for any control law, with no restrictions on the controlled system, such as the minimum-phase condition.

2. THE RLS ALGORITHM IN ADAPTIVE CONTROL

2.1. *The RLS algorithm*

Consider the linear system

$$A(q^{-1};\vartheta^\circ)y(t) = B(q^{-1};\vartheta^\circ)u(t-1)+n(t),\quad t\geq 1, \tag{1a}$$

where $\vartheta^\circ=[a_1^\circ\ a_2^\circ\ \dots\ a_n^\circ\ b_0^\circ\ b_1^\circ\ \dots\ b_{m-1}^\circ]^{\mathrm{T}}\in\mathbb{R}^{n+m}$ is the *system parameter vector* and $A(q^{-1};\vartheta^\circ)$ and $B(q^{-1};\vartheta^\circ)$ are polynomials in the unit delay operator q^{-1} given by

$$A(q^{-1};\vartheta^\circ)=1-\sum_{i=1}^{n}a_i^\circ q^{-i}, \tag{1b}$$

$$B(q^{-1};\vartheta^\circ)=\sum_{i=0}^{m-1}b_i^\circ q^{-i},\quad b_0^\circ\neq 0. \tag{1c}$$

We assume that the transfer funtion of the system is minimal according to the following assumption.

Assumption 1. $q^n A(q^{-1}; \vartheta^\circ)$ and $q^{m-1}B(q^{-1}; \vartheta^\circ)$ are coprime polynomials.

The system (1a) is initialized at time $t = 1$ with given deterministic input and output samples $u(0), \ldots, u(1-m)$, $y(0), \ldots, y(1-n)$. The input $u(t)$, $t \geq 1$, is the control variable, which is a function of past output values (causal control). More precisely, $u(t)$ is any Borel-measurable function of $y(1), y(2), \ldots, y(t)$. The signal $n(t)$ is the equation error subject to the following assumption.

Assumption 2. $\{n(t)\}$ are i.i.d. normally distributed, with $E[n(t)] = 0$ and $E[n(t)^2] = \sigma^2 > 0$.

By introducing the *observation vector*

$$\varphi(t) = [y(t) \quad y(t-1) \quad \ldots \quad y(t-n+1) \quad u(t) \quad u(t-1) \quad \ldots \quad u(t-m+1)]^T, \tag{2}$$

the system (1a) can be given the form

$$y(t) = \varphi(t-1)^T \vartheta^\circ + n(t).$$

The unknown system parameter ϑ° is estimated recursively through the RLS identification algorithm described by the following equations:

$$\hat{\vartheta}(t) = \hat{\vartheta}(t-1) + P(t)\varphi(t-1)[y(t) - \varphi(t-1)^T\hat{\vartheta}(t-1)], \quad \hat{\vartheta}(0) = \hat{\vartheta}_0, \tag{3a}$$

$$P(t) = P(t-1) - \frac{P(t-1)\varphi(t-1)\varphi(t-1)^T P(t-1)}{1 + \varphi(t-1)^T P(t-1)\varphi(t-1)}, \quad P(0) = P_0 = P_0^T > 0. \tag{3b}$$

Note that the auxiliary matrix sequence $P(\cdot)$ is decreasing, so that it tends to some limit as t diverges. This limit will be denoted by $P(\infty)$.

2.2. *Basic properties of the RLS algorithms*

A fundamental issue in the study of RLS consists in pointing out whether or not the estimate $\hat{\vartheta}(t)$ converges and, if so, whether the asymptotic estimate is close to the true parameterization ϑ°. This issue was investigated in the seminal works by Sternby (1977) and Rootzen and Sternby (1984) through the introduction of the so-called Bayesian approach. The basic idea is to take ϑ° as random subject to certain conditions and to show that the RLS equations then coincide with those of the Kalman filter. Since the Kalman filter recursively produces the conditional expectation of ϑ° given the observations, it is then possible to study the RLS algorithm via standard Martingale theory. Using this approach, Rootzen and Sternby proved that the RLS estimate generally converges (Rootzen and Sternby, 1984, Theorem 1) and that the estimation error tends to zero along the directions of diverging information (Rootzen and Sternby, 1984, Theorem 2). Kumar (1990) (see also Chen *et al.*, 1989) remarked that the RLS equations provide the conditional expectation of ϑ° even without requiring any extra integrability condition on the observation vector.

Theorem 1 below summarizes the results on RLS that are relevant to the forthcoming developments in this paper. See Theorem 1 in Kumar (1990) for the proof of point (i) and Theorem 2 in Rootzen and Sternby (1984) for that of point (ii).

Theorem 1. There exists a set $\mathcal{N} \subset \mathbb{R}^{n+m}$ with $\mathcal{L}(\mathcal{N}) = 0$ ($\mathcal{L}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^{n+m}$) such that if $\vartheta^\circ \notin \mathcal{N}$ then

(i) $\lim_{t\to\infty} \hat{\vartheta}(t) = \hat{\vartheta}(\infty)$ a.s., where $\hat{\vartheta}(\infty)$ is an almost surely bounded random variable;

(ii) given an $(n+m)$-dimensional random vector x measurable with respect to $\sigma(y(1), y(2), \ldots)$, we have $\lim_{t\to\infty} x^T\hat{\vartheta}(t) = x^T\vartheta^\circ$, a.s. on $\{P(\infty)x = 0\}$.

Remark 1. The condition $\vartheta^\circ \notin \mathcal{N}$ in Theorem 1 cannot be dropped. As a matter of fact, there are situations in which, if the system parameter vector belongs to a certain singular set with zero Lebesgue measure then the RLS estimate drifts out of any bounded set (Nassiri-Toussi and Ren, 1992). However, this should not be of too much concern: similarly to the way in which almost all the stochastic results hold true with probability one (i.e. for all random occurrences except those belonging to a set of probability measure zero) and still provide a powerful tool in the comprehension of random phenomena, the Bayesian approach helps gain insight into the behaviour of the RLS algorithm even though the corresponding results may fail to hold in a set of Lebesgue measure zero.

In the following, we always assume that $\vartheta^\circ \notin \mathcal{N}$.

The statement of Theorem 1 can be translated into a more suitable form for the forthcoming developments through the use of the so-called excitation subspace, originally introduced in Bittanti *et al.* (1990).

Definition 1. The subspace $\bar{\mathscr{E}} = \{x \in \mathbb{R}^{n+m} \mid (x^T \sum_{t=0}^{\infty} \varphi(t)\varphi(t)^T x < \infty\}$ is called the *unexcitation subspace.* Its orthogonal complement $\mathscr{E} = \bar{\mathscr{E}}^{\perp}$ is called the *excitation subspace.*

Note that $\mathscr{E}$ and $\bar{\mathscr{E}}$ are random subspaces ($\mathscr{E} = \mathscr{E}(\omega)$, $\bar{\mathscr{E}} = \bar{\mathscr{E}}(\omega)$, $\omega \in \Omega$ = basic probability space).

Roughly speaking, the unexcitation subspace is the set of vectors along which the total amount of information remains bounded over time. Since the inverse equation of (3b) is $P(t)^{-1} = P(t-1)^{-1} + \varphi(t-1)\varphi(t-1)^{\mathrm{T}}$, the so-called *algorithm information matrix* $P(t)^{-1}$ turns out to be given by $P(t)^{-1} = P_0^{-1} + \sum_{i=0}^{t-1} \varphi(i)\varphi(i)^{\mathrm{T}}$. Therefore the excitation subspace can also be seen as the null space of the matrix $P(\infty)$.

Theorem 2. Denote by $\hat{\vartheta}_{\mathrm{E}}(t)$ and $\hat{\vartheta}_{\mathrm{U}}(t)$ the projections of $\hat{\vartheta}(t)$ onto $\mathscr{E}$ and $\bar{\mathscr{E}}$ respectively, and by $\vartheta_{\mathrm{E}}^{\circ}$ and $\vartheta_{\mathrm{U}}^{\circ}$ the projections of ϑ° onto $\mathscr{E}$ and $\bar{\mathscr{E}}$ respectively. We have

(i) $\lim_{t\to\infty} \hat{\vartheta}_{\mathrm{U}}(t) = \hat{\vartheta}_{\mathrm{U}}(\infty)$ ($\neq \vartheta_{\mathrm{U}}^{\circ}$, in general) a.s.;

(ii) $\lim_{t\to\infty} \hat{\vartheta}_{\mathrm{E}}(t) = \hat{\vartheta}_{\mathrm{E}}(\infty) = \vartheta_{\mathrm{E}}^{\circ}$ a.s.

Proof. (i) This follows directly from Theorem 1(i).

(ii) Consider the vector $x = \hat{\vartheta}_{\mathrm{E}}(\infty) - \vartheta_{\mathrm{E}}^{\circ}$, which is measurable with respect to $\sigma(y(1), y(2), \ldots)$. Since $x \in \mathscr{E}$, $P(\infty)x = 0$, a.s. . From Theorem 1(ii), we then $\|\hat{\vartheta}_{\mathrm{E}}(\infty) - \vartheta_{\mathrm{E}}^{\circ}\|^2 = x^{\mathrm{T}}(\hat{\vartheta}_{\mathrm{E}}(\infty) - \vartheta_{\mathrm{E}}^{\circ}) = x^{\mathrm{T}}(\hat{\vartheta}(\infty) - \vartheta^{\circ}) = 0$ a.s., which implies $\hat{\vartheta}_{\mathrm{E}}(\infty) = \vartheta_{\mathrm{E}}^{\circ}$ a.s. □

Theorem 2 says that the asymptotic estimation error $\tilde{\vartheta}(\infty) = \hat{\vartheta}(\infty) - \vartheta^{\circ}$ is confined to the unexcitation subspace. This is not surprising, since the estimation error along the excitation directions is compensated by the information, which diverges with time.

2.3. *The pole–zero cancellation problem*

In this section, we address the problem of asymptotic pole–zero cancellation in the estimated transfer function. We first show that, by using the RLS identification algorithm, asymptotic cancellations can actually occur. Then we introduce a modified version of RLS, called PD-RLS ($P(t)$-driven RLS), in order to ensure asymptotic coprimeness. The modification is designed in the light of the results worked out in the previous section. This leads to an identification algorithm that maintains the fundamental property that the estimation error along the excitation directions tends to zero. As discussed in Section 3, when this property is secured, one can develop a stability theory that generally holds for any adaptive control scheme in which the control law is able to stabilize a known invariant system.

We start by providing an example in which RLS leads to an asymptotic cancellation.

Counterexample 1. Take $u(t) = 0\ \forall t$ as the control law and assume that the RLS algorithm is initialized with $P_0 = I$ (the identity matrix) and $\hat{\vartheta}_0$ such that polynomial $q^{m-1}B(q^{-1}; \hat{\vartheta}_0)$ has a common factor with $q^n A(q^{-1}; \vartheta^{\circ})$. It is easy to see that, owing to the presence of noise $n(t)$, the excitation subspace is the subspace spanned by the first n components of the observation vector. Therefore the polynomial $A(q^{-1}; \vartheta^{\circ})$ is consistently estimated (Theorem 2). On the other hand, since $u(t) = 0$, the estimate of $B(q^{-1}; \vartheta^{\circ})$ remains unaltered over time: $B(q^{-1}; \hat{\vartheta}(t)) = B(q^{-1}; \hat{\vartheta}_0)\ \forall t$. Hence $q^n A(q^{-1}; \hat{\vartheta}(\infty))$ and $q^{m-1}B(q^{-1}; \hat{\vartheta}(\infty))$ are not coprime.

For the introduction of the identification algorithm PD-RLS, we need some preliminary definitions.

Given the vector space $\mathbb{R}^{n+m}$ of parameters $\vartheta = [a_1\ a_2\ \ldots\ a_n\ b_0\ b_1\ \ldots\ b_{m-1}]^{\mathrm{T}}$, denote by $\mathscr{C} \subset \mathbb{R}^{n+m}$ the subset of vectors such that $q^n A(q^{-1}; \vartheta)$ and $q^{m-1}B(q^{-1}; \vartheta)$ exhibit a common factor. The set of vectors whose Euclidean distance from $\mathscr{C}$ is less than ϵ is indicated by $\mathscr{C}_{\epsilon}$:

$$\mathscr{C}_{\epsilon} = \{\vartheta : d(\vartheta, \mathscr{C}) < \epsilon\},$$

where $d(\vartheta, \mathscr{C}) := \inf_{\vartheta' \in \mathscr{C}} \|\vartheta - \vartheta'\|$.

Remark 2. The conditions under which two polynomials are coprime is a well-established issue in algebraic geometry. In, for example, Brieskorn and Knörrer (1986, Theorem 3, Section 4.2), it is shown that $\vartheta \in \mathscr{C}$ if and only if the following equation is satisfied:

$$\mathscr{R}(\vartheta) = 0,$$

where $\mathscr{R}(\vartheta)$ is the *resultant* of $q^n A(q^{-1}; \vartheta)$ and $q^{m-1}B(q^{-1}; \vartheta)$, given by

$$\mathscr{R}(\vartheta) = \det \begin{bmatrix} 1 & -a_1 & -a_2 & \ldots & -a_n & & & \\ & 1 & -a_1 & -a_2 & \ldots & -a_n & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & -a_1 & -a_2 & \ldots & -a_n \\ b_0 & b_1 & b_2 & \ldots & b_{m-1} & & & \\ & b_0 & b_1 & b_2 & \ldots & b_{m-1} & & \\ & & \ddots & & & & \ddots & \\ & & & b_0 & b_1 & b_2 & \ldots & b_{m-1} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} m-1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n \end{matrix}$$

In adaptive control, a desired property is that the leading coefficient of the estimated polynomial $B(q^{-1}; \hat{\vartheta}(t))$ is different from zero. In relation to this, we also introduce the set $\mathscr{B}_{\epsilon}$ as the set of vectors whose distance from the hyperplan $\mathscr{B} = \{\vartheta : b_0 = 0\}$ is less than ϵ:

$$\mathscr{B}_\epsilon = \{\vartheta : d(\vartheta, \mathscr{B}) < \epsilon\}.$$

Since $b_0^\circ \neq 0$, and $q^n A(q^{-1}; \vartheta^\circ)$ and $q^{m-1}B(q^{-1}; \vartheta^\circ)$ are coprime polynomials (Assumption 1), the true parameterization ϑ° does not belong to $\mathscr{C}_\epsilon \cup \mathscr{B}_\epsilon$ for sufficiently small ϵ. We assume that some a priori information on ϑ° is available, so that we can fix an $\bar{\epsilon} > 0$ such that ϑ° belongs to the interior of the complement of $\mathscr{C}_{2\bar{\epsilon}} \cup \mathscr{B}_{2\bar{\epsilon}}$.

The PD-RLS algorithm is as follows.

P(t)-driven RLS algorithm (PD-RLS).

$$\hat{\vartheta}(t) = \hat{\vartheta}(t-1) + P(t)\varphi(t-1)[y(t) - \varphi(t-1)^{\mathrm{T}}\hat{\vartheta}(t-1)], \quad \hat{\vartheta}(0) = \hat{\vartheta}_0,$$

$$P(t) = P(t-1) - \frac{P(t-1)\varphi(t-1)\varphi(t-1)^{\mathrm{T}}P(t-1)}{1 + \varphi(t-1)^{\mathrm{T}}P(t-1)\varphi(t-1)}, \quad P(0) = P_0 = P_0^{\mathrm{T}} > 0.$$

Recursively construct a vector sequence $v(\cdot) \in \mathbb{R}^{n+m}$ with $v(t)$ depending upon $\hat{\vartheta}(\tau)$ and $P(\tau)$, $\tau \leq t$, such that

(i) $v(\cdot)$ a.s. convergent;

(ii) $\hat{\vartheta}(t) + P(t)v(t) \in$ complement $(\mathscr{C}_{\bar{\epsilon}} \cup \mathscr{B}_{\bar{\epsilon}})\ \forall t$ a.s.

The PD-RLS estimate is given by

$$\hat{\vartheta}'(t) = \hat{\vartheta}(t) + P(t)v(t).$$

Remark 3. The above procedure defines a family of PD-RLS algorithms, each of which given by a different rule for the determination of the perturbation vector $v(\cdot)$. The proof that $v(\cdot)$ sequences such that conditions (i) and (ii) actually exist is provided in Theorem 4 below.

Remark 4. Contrary to most algorithms proposed in the literature (see e.g. Praly *et al.*, 1989; Wen and Hill, 1992), PD-RLS does not incorporate any projection operators. This is crucial in order to establish the fundamental properties of the algorithm stated in Theorem 3.

Remark 5. The idea of using perturbation of the RLS estimate of the kind proposed herein is not new. In a deterministic setting, Lozano and Zaho (1994) introduced a similar modification, and proved that it provided controllable models in a pole-placement context. In contrast to the stochastic method introduced in the present contribution, the algorithm in Lozano and Zaho (1994) is inherently based on the assumption that the noise is deterministically bounded by a known quantity. This allows one to construct a suitable normalized least-squares algorithm (with a dead zone depending on the disturbance bound) in which the perturbation vector is selected within a finite set of possible values. Interestingly enough, such a method is too stiff in a stochastic framework, where the unboundedness of noise calls for a different line of reasoning, as provided in the present paper.

The modification introduced in the PD-RLS algorithm with respect to the standard RLS has a twofold objective:

- to keep the estimate away from the 'dangerous' set $\mathscr{C}$ where polynomials $q^n A(q^{-1}; \vartheta)$ and $q^{m-1}B(q^{-1}; \vartheta)$ are not coprime;
- to preserve the fundamental property that the estimation error tends to zero along the excitation directions.

The first of these is of great importance, since it implies the coprimeness of the asymptotically estimated transfer function (see Theorem 3). On the other hand, when the algorithm is used in an adaptive control scheme, the second requirement plays a crucial role in securing the stability property and the overall control system performance. This assertion can be made intuitively clear by noting that the observation vector tends to zero along the unexcitation directions (see Definition 1). Therefore the behavior of the controlled system in the long run is determined only by the component of the parameter vector in the excitation subspace, and accurate knowledge of this component is sufficient to guarantee the desired performance. This idea will become more concrete in the next section, when we address the question of adaptive stabilization.

In PD-RLS, the above two objectives are pursued by means of the extra term $P(t)v(t)$ in the estimation equation. On the one hand, this term forces the estimation error to lie in the 'safe' region in which no pole–zero cancellation occurs. On the other hand, since $P(t)$ vanishes in the excitation directions, this modification is expected not to hinder the RLS partial consistency property along these directions. These considerations are given solid bases in the following theorem.

Theorem 3. The PD-RLS estimate $\hat{\vartheta}'(t)$ is almost surely convergent to a bounded random variable $\hat{\vartheta}'(\infty)$ such that $\hat{\vartheta}'_{\mathrm{E}}(\infty) = \vartheta^\circ_{\mathrm{E}}$ (where $\hat{\vartheta}'_{\mathrm{E}}(\infty)$ and $\vartheta^\circ_{\mathrm{E}}$ are the projections of $\hat{\vartheta}'(\infty)$ and ϑ° respectively onto $\mathscr{E}$. Moreover, $q^n A(q^{-1}; \hat{\vartheta}'(\infty))$ and $q^{m-1}B(q^{-1}; \hat{\vartheta}'(\infty))$ are almost surely coprime polynomials and $\hat{b}'_0(\infty)$ (the asymptotic estimate of b_0°) is almost surely different from zero.

Proof. The convergence of $\hat{\vartheta}'(t)$ follows directly from the convergence of $\hat{\vartheta}(t)$ (Theorem 1), from the fact that the matrix $P(t)$ is monotonically decreasing and therefore convergent, and from the convergence of $v(t)$ (point (i) in the definition of the PD-RLS algorithm). The relation $\hat{\vartheta}'_E(\infty) = \vartheta^\circ_E$ is a consequence of the fact that the matrix $P(t)$ tends to zero along the excitation directions, so that, along these directions, $\hat{\vartheta}'(t) - \hat{\vartheta}(t) = P(t)v(t) \to 0$. Finally, since by construction the estimate $\hat{\vartheta}'(t)$ keeps away from $\mathscr{C} \cup \mathscr{B}$ by at least a distance $\bar{\epsilon}$, we have $q^n A(q^{-1}; \hat{\vartheta}'(\infty))$ and $q^{m-1}B(q^{-1}; \hat{\vartheta}'(\infty))$ a.s. coprime and $\hat{b}'_0(\infty)$ a.s. different from zero.□

In the next theorem, we provide a proof of the existence of PD-RLS algorithms.

Theorem 4. The class of PD-RLS algorithms is not empty.

Proof. Let $V(t) = \{v : \hat{\vartheta}(t) + P(t)v \in \text{complement}\ (\mathscr{C}_{2\bar{\epsilon}} \cup \mathscr{B}_{2\bar{\epsilon}})\}$. Obviously, $V(t) \neq \emptyset$. We first prove that

$$\alpha(t) = \inf\{\|v\|,\ v \in V(t)\} \text{ is bounded a.s.}$$

Set $\bar{v} = P(\infty)^\dagger[\vartheta^\circ - \hat{\vartheta}(\infty)]$ (where $P(\infty)^\dagger$ denotes the pseudoinverse of $P(\infty)$). We have

$$\begin{aligned}
\hat{\vartheta}(t) + P(t)\bar{v} &= \hat{\vartheta}(t) + P(\infty)\bar{v} + [P(t) - P(\infty)]\bar{v} \\
&= \hat{\vartheta}(t) + [\vartheta^\circ - \hat{\vartheta}(\infty)] + [P(t) - P(\infty)]\bar{v} \\
&\quad (\text{since } \vartheta^\circ - \hat{\vartheta}(\infty) \in \text{image}\,[P(\infty)]) \\
&= \vartheta^\circ + [\hat{\vartheta}(t) - \hat{\vartheta}(\infty)] + [P(t) - P(\infty)]\bar{v} \\
&\to \vartheta^\circ \quad \text{a.s.}
\end{aligned}$$

Since ϑ° is in the interior of the complement of $\mathscr{C}_{2\bar{\epsilon}} \cup \mathscr{B}_{2\bar{\epsilon}}$, there exists a time $\bar{t}$ such that, $\forall t > \bar{t}$, $\bar{v} \in V(t)$, from which the a.s. boundedness of $\alpha(t)$ follows.

On the grounds of the above boundedness result, we can now show a possible construction of the sequence $v(\cdot)$.

For $t = 0$ pick a $v(0) \in V(0)$ such that $\|v(0)\| = \alpha(0)$.

For $t \geq 1$,

if $\hat{\vartheta}(t) + P(t)v(t-1) \in \text{complement}\ (\mathscr{C}_{\bar{\epsilon}} \cup \mathscr{B}_{\bar{\epsilon}})$,

$$v(t) = v(t-1)$$

otherwise, pick a $v(t) \in V(t)$

such that $\|v(t)\| = \alpha(t)$.

Obviously, $v(\cdot)$ satisfies condition (ii) in the definition of PD-RLS. To see that $v(\cdot)$ is a.s. convergent (condition (i)), note first that from the relation $\sup_t \|v(t)\| \leq \sup_t \alpha(t)$ and the boundedness of $\alpha(\cdot)$, it follows that $\|v(\cdot)\|$ is a.s. bounded. Set $b_v := \sup_t \|v(t)\|$. Then, taking into account that $\hat{\vartheta}(t)$ and $P(t)$ are convergent sequences, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{\tau \geq t} \|[\hat{\vartheta}(\tau) + P(\tau)v(t)] - [\hat{\vartheta}(t) + P(t)v(t)]\| \\
&\quad \leq \limsup_{t\to\infty} \sup_{\tau \geq t} [\|\hat{\vartheta}(\tau) - \hat{\vartheta}(t)\| + \|P(\tau) - P(t)\| b_v] \\
&\quad = 0 \quad \text{a.s.} \qquad (4)
\end{aligned}$$

By the definition of $V(t)$, at the instants at which a new $v(t)$ is selected, we have $\hat{\vartheta}(t) + P(t)v(t) \in \text{complement}\ (\mathscr{C}_{2\bar{\epsilon}} \cup \mathscr{B}_{2\bar{\epsilon}})$, and a new $v(\tau)$, $\tau > t$, is successively chosen only if $\hat{\vartheta}(\tau) + P(\tau)v(t) \notin \text{complement}\ (\mathscr{C}_{\bar{\epsilon}} \cup \mathscr{B}_{\bar{\epsilon}})$ (compare $\mathscr{C}_{2\bar{\epsilon}} \cup \mathscr{B}_{2\bar{\epsilon}}$ with $\mathscr{C}_{\bar{\epsilon}} \cup \mathscr{B}_{\bar{\epsilon}}$). Therefore it follows from (4) that the number of instants at which a new selection actually occurs is finite. The convergence of $v(\cdot)$ follows from this. □

In the proof, a sequence $v(\cdot)$ that satisfies requirements (i) and (ii) in the definition of the algorithm PD-RLS is introduced. We remark now that there is no reason to believe that this choice is in any sense optimal. Therefore an appropriate selection rule of the sequence $v(\cdot)$ so as to minimize the computational effort of the algorithm, and possibly to meet *ad hoc* requirements, should be the subject of further investigation.

3. APPLICATION EXAMPLE: ADAPTIVE STABILIZATION

The adaptive stabilization of a plant with unknown parameters has long been studied. Not only is this issue important by itself, but it often constitutes a first fundamental step in the performance analysis of an adaptive control scheme. Most papers on the subject deal with specific control laws, so that a plethora of *ad hoc* stability results can be found; see e.g. Goodwin *et al.* (1981), Sin and Goodwin (1982), Bittanti *et al.* (1992) and Campi (1992) for the minimum-variance regulator, Guo and Chen (1991) for the Åström–Wittenmark regulator, and Ren (1993) for a pole-placement controller. Kumar (1990) also provided a general result, which, however, is applicable to minimum-phase systems only.

Our goal in the present section is to present a stability result of general validity. It holds true regardless of the specific control law and without requiring restrictive assumptions on the controlled system, such as the minimum-phase condition. To be specific, consider the pathwise mean stability condition

$$\lim_{N\to\infty} \sup \frac{1}{N} \sum_{t=1}^{N} [|u(t)| + |y(t)|] < \infty \quad \text{a.s.} \qquad (5)$$

We show that any control law able to stabilize a known invariant system meets the condition (5) when used in an adaptive fashion, provided that the identification is performed through the PD-RLS algorithm. The generality of such a result makes it applicable without restriction to any adaptive scheme in which the PD-RLS algorithm is used.

Consider the following invariant plant:

$$A(q^{-1}; \bar{\vartheta})y(t) = B(q^{-1}; \bar{\vartheta})u(t-1), \quad (6a)$$

where the polynomials $A(q^{-1}; \bar{\vartheta})$ and $B(q^{-1}; \bar{\vartheta})$ are defined similarly to (1b, c):

$$A(q^{-1}; \bar{\vartheta}) = 1 - \sum_{i=1}^{n} \bar{a}_i q^{-i}, \quad (6b)$$

$$B(q^{-1}; \bar{\vartheta}) = \sum_{i=0}^{m-1} \bar{b}_i q^{-i}, \quad \bar{b}_0 \neq 0. \quad (6c)$$

For the *known* system (6), many well-established techniques that exhibit stabilizing properties provided that the system has no pole–zero cancellations (i.e. $q^n A(q^{-1}; \bar{\vartheta})$ and $q^{m-1}B(q^{-1}; \bar{\vartheta})$ are coprime polynomials) are available. Among others, we mention infinite-horizon LQ control (Anderson and Moore, 1989), pole placement (Åström and Wittenmark, 1980) and receding-horizon control (Mosca and Zhang, 1992; Chisci and Mosca, 1993). All these techniques lead to control laws of the form

$$u(t) = R(q^{-1}; \bar{\vartheta})u(t) + S(q^{-1}; \bar{\vartheta})y(t), \quad (7)$$

where

$$R(q^{-1}; \vartheta) = \sum_{i=1}^{\alpha} r_i(\vartheta) q^{-i},$$

$$S(q^{-1}; \vartheta) = \sum_{i=0}^{\beta} s_i(\vartheta) q^{-i},$$

and the coefficients $r_i(\vartheta)$ and $s_i(\vartheta)$ are continuous functions of ϑ in a neighborhood of the true parameterization $\bar{\vartheta}$. The stability property can be stated precisely as follows. Letting $z(t) = [y(t) \;\; u(t)]^T$, the system (6) + (7) can be written as

$$z(t) = D(q^{-1}; \bar{\vartheta})z(t-1), \quad (8)$$

where the polynomial matrix $D(q^{-1}; \vartheta)$ is defined as

$$D(q^{-1}; \vartheta) = \begin{bmatrix} [1 - A(q^{-1}; \vartheta)]q & B(q^{-1}; \vartheta) \\ S(q^{-1}; \vartheta)q & R(q^{-1}; \vartheta)q \end{bmatrix}.$$

The stability condition is then written as $\|z(t)\| \le \alpha\rho^{\Delta t} \|\bar{z}(t_0)\|$, $\rho < 1$, $\Delta t = t - t_0$, where $\bar{z}(t)$ is the state of the system (8) given by

$$\bar{z}(t) = [y(t) \;\; \ldots \;\; y(t+1-\max\{n, \beta\}) \;\; u(t) \;\; \ldots \;\; u(t+1-\max\{m, \alpha\})]^T.$$

Let us suppose now that the control law (7) is applied in a certainty-equivalent fashion to our true system (1a). The actual adaptive control law is then given by

$$u(t) = R(q^{-1}; \hat{\vartheta}'(t))u(t) + S(q^{-1}; \hat{\vartheta}'(t))y(t), \quad (9)$$

where $\hat{\vartheta}'(t)$ is computed through the PD-RLS algorithm. Then the *overall control* (OC) system (1a) + (9) can be written in matrix form as

$$z(t) = D(q^{-1}; \hat{\vartheta}'(t))z(t-1) + \begin{bmatrix} e(t) \\ 0 \end{bmatrix}, \quad (10)$$

with

$$e(t) = \varphi(t-1)^T[\vartheta^\circ - \hat{\vartheta}'(t)] + n(t).$$

The system $z(t) = D(q^{-1}; \hat{\vartheta}'(t))z(t-1)$ is the *overall estimated* (OE) system at time t, and $e(t)$ is a perturbation term that accounts for the additive noise $n(t)$ and, more importantly, for the discrepancy between the true system and the estimated one.

We now want to prove that if the control law is selected among the above described set of stabilizing techniques then the OE system is almost surely uniformly exponentially stable.

Note that the *overall asymptotically estimated* (OAE) system $z(t) = D(q^{-1}; \hat{\vartheta}'(\infty))z(t-1)$ is almost surely exponentially stable because the polynomials $q^n A(q^{-1}; \hat{\vartheta}'(\infty))$ and $q^{m-1}B(q^{-1}; \hat{\vartheta}'(\infty))$ are almost surely coprime (Theorem 3). Therefore $\|z(t)\| \le \alpha\rho^{\Delta t} \|\bar{z}(t_0)\|$, $\rho < 1$, $\Delta t = t - t_0$, where $z(t)$ is generated by the OAE system (α and ρ actually depend upon the outcome $\omega \in \Omega$). Taking into account that $\hat{\vartheta}'(t) \to \hat{\vartheta}'(\infty)$ and that $D(q^{-1}; \vartheta)$ is almost surely continuous in $\hat{\vartheta}'(\infty)$, for the OE System we then have $\|z(t)\| \le \alpha\rho^{\Delta t} \|\bar{z}(t_0)\| + \gamma(t_0, \Delta t) \|\bar{z}(t_0)\|$, where $\gamma(\cdot, \cdot)$ is such that $\gamma(t_0, \Delta t) \to 0$, $t_0 \uparrow \infty$, Δt fixed. Choose $\Delta\bar{t}$ such that $\alpha\rho^{\Delta\bar{t}} = \beta < 1$ and $\bar{t}_0$ such that $|\gamma(t_0, \Delta\bar{t})| \le \delta < 1 - \beta$, $\forall t_0 \ge \bar{t}_0$. Then, $\forall t_0 \ge \bar{t}_0$, one has $\|z(t_0 + \Delta\bar{t})\| \le (\beta + \delta) \|\bar{z}(t_0)\|$, from which the uniform exponential stability of the OE system follows.

Now consider the OC system (10). In view of the uniform exponential stability of the OE system, $z(t)$ generated by the OC system can be bounded as follows:

$$\|z(t)\| \le c_1 + c_2 \sum_{i=1}^{t} \nu^{t-i} \|e(i)\|$$

$$\le c_1 + c_2 \sum_{i=1}^{t} \nu^{t-i}\{|\varphi(i-1)^T[\vartheta^\circ - \hat{\vartheta}'(i)]| + |n(i)|\}, \quad \text{a.s.},$$

c_1, c_2 and $\nu < 1$ being suitable constants (depending upon $\omega \in \Omega$). Bearing in mind the definition of the observation vector given in (2), from the previous inequality one obtains

$$\frac{1}{N}\sum_{t=1}^{N}\|\varphi(t)\| \leq c_3 + c_4\frac{1}{N}\sum_{i=1}^{N}|\varphi(i-1)^{\mathrm{T}}[\vartheta^\circ - \hat{\vartheta}'(i)]| + c_4\frac{1}{N}\sum_{i=1}^{N}|n(i)|, \quad \text{a.s.,} \qquad (11)$$

with c_3 and c_4 suitable constants. Obviously, $(1/N)\sum_{i=1}^{N}|n(i)|$ is almost surely bounded. As for the second term on the right-hand side, by decomposing $\varphi(i-1)$ and $\vartheta^\circ - \hat{\vartheta}'(i)$ into their $\mathscr{E}$ and $\bar{\mathscr{E}}$ components, it can be handled as follows:

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}&|\varphi(i-1)^{\mathrm{T}}[\vartheta^\circ - \hat{\vartheta}'(i)]| \\ &\leq \frac{1}{N}\sum_{i=1}^{N}\{|\varphi_{\mathrm{E}}(i-1)^{\mathrm{T}}[\vartheta_{\mathrm{E}}^\circ - \hat{\vartheta}'_{\mathrm{E}}(i)]| + |\varphi_{\mathrm{U}}(i-1)^{\mathrm{T}}[\vartheta_{\mathrm{U}}^\circ - \hat{\vartheta}'_{\mathrm{U}}(i)]|\} \\ &\leq \frac{1}{N}\sum_{i=1}^{N}\|\varphi_{\mathrm{E}}(i-1)\|\,\|\vartheta_{\mathrm{E}}^\circ - \hat{\vartheta}'_{\mathrm{E}}(i)\| + \frac{1}{N}\sum_{i=1}^{N}\|\varphi_{\mathrm{U}}(i-1)\|\,\|\vartheta_{\mathrm{U}}^\circ - \hat{\vartheta}'_{\mathrm{U}}(i)\|.\end{aligned}$$

The second term in this last expression tends to zero because of the boundedness of $\|\vartheta_{\mathrm{U}}^\circ - \hat{\vartheta}'_{\mathrm{U}}(i)\|$ and the fact that $\varphi_{\mathrm{U}}(i-1) \to 0$ as $i \to \infty$ (see the definition of the unexcitation subspace). Therefore, from (11), we finally get

$$\begin{aligned}\frac{1}{N}\sum_{t=1}^{N}&\|\varphi(t)\| \\ &\leq c_5 + c_6\frac{1}{N}\sum_{i=1}^{N}\|\varphi_{\mathrm{E}}(i-1)\|\,\|\vartheta_{\mathrm{E}}^\circ - \hat{\vartheta}'_{\mathrm{E}}(i)\| \\ &\leq c_5 + c_6\frac{1}{N}\sum_{i=1}^{N}\|\varphi(i-1)\|\,\|\vartheta_{\mathrm{E}}^\circ - \hat{\vartheta}'_{\mathrm{E}}(i)\|, \quad \text{a.s.,}\end{aligned}$$

with c_5 and c_6 suitable constants. Since $\vartheta_{\mathrm{E}}^\circ - \hat{\vartheta}'_{\mathrm{E}}(i) \to 0$ a.s. (Theorem 3), this inequality entails that $(1/N\sum_{t=1}^{N}\|\varphi(t)\|$ remains almost surely bounded. This, in turn, implies the result (5).

The stability result derived in this section is summarized in the following theorem.

Theorem 5. Suppose that the system (1) is adaptively controlled through (9) and that the non-adaptive version of the control law (9) is able to stabilize known time-invariant controllable systems (i.e. (6) + (7) is a stable system whenever $q^nA(q^{-1};\bar{\vartheta})$ and $q^{m-1}B(q^{-1};\bar{\vartheta})$ are coprime). Then the system (1) + (9) is pathwise mean stable, i.e. (5) is satisfied.

4. CONCLUSIONS

As is well known, the controllability of the estimated model plays a crucial role in adaptive control. In this paper, we have shown that, irrespective of the control law used, such a condition is achieved with probability one provided that a suitable variant of the least-squares algorithm is used in the parameter identification. The proposed method preserves all the basic properties of the standard least-squares algorithm. In particular, the corresponding estimation error tends to zero in the directions of diverging information, which has been proved fundamental in order to guarantee general stabilizing properties of adaptive control systems.

The analysis of the present paper is tailored on the recursive least-squares algorithm. As such, it is not easily extendible to stochastic approximation methods. It would be of considerable interest to work out analogous results for this important class of techniques.

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