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Adaptive pole placement by means of a simple, singularity free, identification algorithm

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Abstract

Reportedly, guaranteeing the controllability of the estimated system is a crucial problem in adaptive control. In this paper, we introduce a recursive least squares-based identification algorithm for stochastic SISO systems, which secures the uniform controllability of the estimated system and presents closed-loop identification properties similar to those of the least squares algorithm. The proposed algorithm is recursive and, therefore, easily implementable. Its use, however, is confined to cases in which the parameter uncertainly is highly structured.

This new identification algorithm can be safely used in adaptive control applications. As a matter of fact, we introduce a pole placement adaptive control scheme equipped with such an algorithm and prove a pathwise stability result for the so-obtained closed-loop system.

1 Introduction

In this paper, we introduce a new recursive least squares-based identification algorithm to cope with the long-standing controllability problem in adaptive control. As a matter of fact, it is well known ([1]-[7]) that the possible occurrence of pole-zero cancellations in the estimated model hampers the operation of adaptive control systems when the plant is nonminimum-phase. On the other hand, in the absence of suitable identifiability conditions, standard identification algorithms do not guarantee the estimated model controllability.

Two main streams of methods have been proposed in the literature to solve the controllability problem. One consists in the a-posteriori modification of the least squares estimate ([2]-[4]). By exploiting the properties of the least squares covariance matrix, these methods secure controllability, while preserving the least squares algorithm properties. The main drawback of this approach is that the modification is not easily implementable. The second approach ([5]-[7]) forces the estimates to belong to an a-priori known region containing the true parameter and such that all the models in that region are controllable.

The solution we propose in this paper belongs to the second group of approaches briefly described above. The required a-priori knowledge is certainly a restrictive assumption, but, in the case such a knowledge is in fact available, the identification algorithm we propose represents an efficient and easily implementable way to circumvent the controllability problem. Moreover, we show that our modification to force the estimate to belong to the known uncertainty region is active only in finite time and it switches off automatically in the long run. As a consequence, the new identification method retains the closed-loop identification properties of the standard least squares method (Theorem 2 in Section 3). This is of crucial importance in adaptive control applications.

The final section (Section 4) is dedicated to the application of the proposed identification method to an adaptive pole placement control problem. We prove in particular a pathwise stability result for the corresponding control scheme.

2 The system and the uncertainty region

We consider the discrete time stochastic SISO system described by the following ARX model

$$A(\vartheta^\circ;q^{-1})\,y_t = B(\vartheta^\circ;q^{-1})\,u_t + n_t, \qquad (1)$$

where $A(\vartheta^\circ;q^{-1})$ and $B(\vartheta^\circ;q^{-1})$ are polynomials in the unit-delay operator q^{-1} depending on the system parameter vector $\vartheta^\circ = [a_1^\circ\ a_2^\circ \dots a_n^\circ\ b_d^\circ\ b_{d+1}^\circ \dots b_{m+d}^\circ]^T$. Precisely, they are given by $A(\vartheta^\circ;q^{-1}) = 1 - \sum_{i=1}^n a_i^\circ q^{-i}$ and $B(\vartheta^\circ;q^{-1}) = \sum_{i=d}^{m+d} b_i^\circ q^{-i}$.

As for the stochastic disturbance $\{n_t\}$ acting on the system, it is described as a martingale difference sequence with respect to an increasing sequence of σ-fields $\{\mathcal{F}_t\}$, satisfying the following conditions

A.1) $\sup_t E[|n_{t+1}|^\beta/\mathcal{F}_t] < \infty$, a.s. for some $\beta > 2$,

A.2) $\liminf_{t\to\infty} \frac{1}{t}\sum_{k=1}^t n_k^2 > 0$.

In this paper, a new identification algorithm for system (1) is introduced, which secures the estimated model controllability, while preserving the least squares algorithm closed-loop identification properties. These results can be worked out under the a-priori knowledge that

A.3) ϑ° is an interior point of $S(\bar{\vartheta}, r) = \{\vartheta \in R^{n+m+1} : \|\vartheta - \bar{\vartheta}\| \leq r\}$,

where the $n+m+1$-sphere $S(\bar{\vartheta}, r)$ is such that all models with parameter $\vartheta \in S(\bar{\vartheta}, r)$ are controllable. Assumption 3 is certainly a stringent condition. It requires that the a-priori parameter uncertainty is restricted enough so that the uncertainty region can be described as a sphere completely embedded in the controllability region. In this connection, the center $\bar{\vartheta}$ of the sphere should be thought of as a nominal, a-priori known, value of the uncertain parameter ϑ°, obtained either by physical knowledge of the plant or by some coarse off-line identification procedure. The identification algorithm should then be used to refine the parameter estimate during the normal on-line operating condition of the control system so as to better tune the controller to the actual plant characteristics.

3 The recursive identification algorithm

Letting $\varphi_t = [y_t \cdots y_{t-(n-1)}\ u_{t-(d-1)} \cdots u_{t-(m+d-1)}]^T$ be the observation vector, system (1) can be given the usual regression-like form

$$y_t = \varphi_{t-1}^T \vartheta^\circ + n_t.$$

The recursive algorithm for the estimation of parameter ϑ° is given by the following recursive procedure:

1. Compute P_t according to the following procedure:

   set $T_0 = P_{t-1}$

   for $i = 1$ to $n+m+1$,

   $$\phi_i = [0 \ldots \underset{\substack{\uparrow \\ i}}{1} \ldots 0]^T$$

   $$T_i = T_{i-1} - \frac{(\alpha_t - \alpha_{t-1}) T_{i-1} \phi_i \phi_i^T T_{i-1}}{1 + (\alpha_t - \alpha_{t-1}) \phi_i^T T_{i-1} \phi_i}$$

   then,

   $$P_t = T_{n+m+1} - \frac{T_{n+m+1} \varphi_{t-1} \varphi_{t-1}^T T_{n+m+1}}{1 + \varphi_{t-1}^T T_{n+m+1} \varphi_{t-1}}. \quad (2.1)$$

2. Compute the least squares type estimate $\hat{\vartheta}_t$ according to the equation:

   $$\hat{\vartheta}_t = \bar{\vartheta}_{t-1} + P_t \varphi_{t-1}(y_t - \varphi_{t-1}^T \bar{\vartheta}_{t-1}) + P_t(\alpha_t - \alpha_{t-1})(\bar{\vartheta} - \bar{\vartheta}_{t-1}) \quad (2.2)$$

   where

   $$r_t = r_{t-1} + \|\varphi_{t-1}\|^2 \quad (2.3)$$

   $$\alpha_t = (\log(r_t))^{1+\delta},\ (\delta > 0). \quad (2.4)$$

3. If $\hat{\vartheta}_t \notin S(\bar{\vartheta}, r)$, project the estimate $\hat{\vartheta}_t$ onto the sphere $S(\bar{\vartheta}, r)$:

   $$\bar{\vartheta}_t = \begin{cases} \hat{\vartheta}_t, & \text{if } \hat{\vartheta}_t \in S(\bar{\vartheta}, r) \\ \frac{\hat{\vartheta}_t - \bar{\vartheta}}{\|\hat{\vartheta}_t - \bar{\vartheta}\|} r + \bar{\vartheta}, & \text{otherwise.} \end{cases} \quad (2.5)$$

It is possible to verify (Theorem 1 below) that equations (2.1)-(2.4) recursively compute the minimizer of the performance index

$$\sum_{k=1}^{t} (y_k - \varphi_{k-1}^T \vartheta)^2 + \alpha_t \|\vartheta - \bar{\vartheta}\|^2 \quad (3)$$

(here we neglect the initialization issue, see Theorem 1 below for a precise statement). This observation allows for an easy interpretation of the algorithm (2.1)-(2.5). In equation (3), the first term $\sum_{k=1}^{t} (y_k - \varphi_{k-1}^T \vartheta)^2$ is the standard performance index for the least squares algorithm, while the second term $\alpha_t \|\vartheta - \bar{\vartheta}\|^2$ penalizes those parameterizations which are too far from the a-priori nominal parameter value $\bar{\vartheta}$. In Theorem 1 point *ii)*, we show that the coefficient α_t in front of $\|\vartheta - \bar{\vartheta}\|^2$ grows rapidly enough so that term $\alpha_t \|\vartheta - \bar{\vartheta}\|^2$ asserts itself in the long run in such a way that the estimate $\hat{\vartheta}_t$ belongs to $S(\bar{\vartheta}, r)$, for t large enough. As a consequence, the projection operator in equation (2.5) is automatically switched off when t tends to infinity. The fact that the estimate becomes free of any projection in the long run has a beneficial effect on its asymptotic properties. As a matter of fact, in Theorem 2 we prove that $\bar{\vartheta}_t$ exhibits closed-loop properties which are similar to those of the standard recursive least squares estimate. This is crucial in adaptive control applications.

Theorem 1
i) The parameter estimate $\hat{\vartheta}_t$ obtained through the recursive procedure (2.1)-(2.4) initialized with

$$\begin{cases} \hat{\vartheta}_0 = \bar{\vartheta} \\ r_0 = tr(Q) \\ \alpha_0 = (\log(r_0))^{1+\delta} \\ P_0 = [Q + \alpha_0 I]^{-1} \end{cases} \quad (Q = Q^T > 0)$$

is the minimizer of the performance index

$$D_t(\vartheta) = V_t(\vartheta) + \alpha_t \|\vartheta - \bar{\vartheta}\|^2$$

where

$$V_t(\vartheta) = \sum_{k=1}^{t} (y_k - \varphi_{k-1}^T \vartheta)^2 + (\vartheta - \bar{\vartheta})^T Q (\vartheta - \bar{\vartheta})$$

is the standard least squares performance index with regularization term $(\vartheta - \bar{\vartheta})^T Q (\vartheta - \bar{\vartheta})$ and

$$\alpha_t = (\log(\sum_{k=1}^{t} \|\varphi_{k-1}\|^2 + tr(Q)))^{1+\delta}. \quad (4)$$

ii) Assume that u_t is $\mathcal{F}_t$-measurable. Then, there exists a finite time instant $\bar{t}$ such that $\hat{\vartheta}_t \in S(\bar{\vartheta}, r)$, $t \geq \bar{t}$, a.s..

Proof.
Part i) Trivial and therefore omitted.

Part ii) Denote by $\hat{\vartheta}_t^{LS}$ the minimizer of the least squares performance index $V_t(\vartheta)$ and set

$$Q_t = \sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q.$$

It is then easy to show that $\hat{\vartheta}_t = arg \min_{\vartheta \in R^{n+m+1}} D_t(\vartheta)$ can be expressed as a function of $\hat{\vartheta}_t^{LS}$ as follows

$$\hat{\vartheta}_t = (Q_t + \alpha_t I)^{-1} Q_t \hat{\vartheta}_t^{LS} + \alpha_t (Q_t + \alpha_t I)^{-1} \bar{\vartheta}.$$

By subtracting $\bar{\vartheta}$, we get

$$\hat{\vartheta}_t - \bar{\vartheta} = (Q_t + \alpha_t I)^{-1} Q_t (\vartheta^\circ - \bar{\vartheta}) + (Q_t + \alpha_t I)^{-1} Q_t (\hat{\vartheta}_t^{LS} - \vartheta^\circ).$$

Thus, the norm of $\hat{\vartheta}_t - \bar{\vartheta}$ can be upper bounded as follows

$$\|\hat{\vartheta}_t - \bar{\vartheta}\| \leq \|\vartheta^\circ - \bar{\vartheta}\| + \|(Q_t + \alpha_t I)^{-1} Q_t^{\frac{1}{2}}\| \|Q_t^{\frac{1}{2}}(\hat{\vartheta}_t^{LS} - \vartheta^\circ)\|. \quad (5)$$

We apply now Theorem 1 in reference [8] so as to upper bound the term $\|Q_t^{\frac{1}{2}}(\hat{\vartheta}_t^{LS} - \vartheta^\circ)\|$. Since u_t is assumed to be $\mathcal{F}_t$-measurable, and also considering Assumption 1, by this theorem we obtain the upper bound on the least squares estimation error:

$$\|Q_t^{\frac{1}{2}}(\hat{\vartheta}_t^{LS} - \vartheta^\circ)\|^2 = O(\log(tr(Q_t))), \text{ a.s..} \quad (6)$$

The term $\|(Q_t + \alpha_t I)^{-1} Q_t^{\frac{1}{2}}\|$ can instead be handled as follows. Denote by $\{\lambda_{1,t}, \ldots, \lambda_{n+m+1,t}\}$ the eigenvalues of the positive definite matrix Q_t. Since Q_t is symmetric and positive definite, there exists an orthonormal matrix T_t such that $Q_t = T_t \; diag(\lambda_{1,t}, \ldots, \lambda_{n+m+1,t}) T_t^{-1}$ and $Q_t^{\frac{1}{2}} = T_t \; diag\left(\lambda_{1,t}^{\frac{1}{2}}, \ldots, \lambda_{n+m+1,t}^{\frac{1}{2}}\right) T_t^{-1}$. Then,

$$(Q_t + \alpha_t I)^{-1} Q_t^{\frac{1}{2}} = T_t (T_t^{-1}(Q_t + \alpha_t I) T_t)^{-1} T_t^{-1} Q_t^{\frac{1}{2}} T_t T_t^{-1}$$

$$= T_t \; diag\left(\frac{\lambda_{1,t}^{\frac{1}{2}}}{\lambda_{1,t} + \alpha_t}, \ldots, \frac{\lambda_{n+m+1,t}^{\frac{1}{2}}}{\lambda_{n+m+1,t} + \alpha_t}\right) T_t^{-1}.$$

This implies that

$$\|(Q_t + \alpha_t I)^{-1} Q_t^{\frac{1}{2}}\| = \max_{i=1,\ldots,n+m+1} \left(\frac{\lambda_{i,t}^{\frac{1}{2}}}{\lambda_{i,t} + \alpha_t}\right). \quad (7)$$

Consider now the function: $f(x) = \dfrac{x^{\frac{1}{2}}}{x + \alpha_t}, x \geq 0$. Such a function has an absolute maximum value $\frac{1}{2}\alpha_t^{-\frac{1}{2}}$ in $x = \alpha_t$. It then obviously follows from equation (7) that

$$\|(Q_t + \alpha_t I)^{-1} Q_t^{\frac{1}{2}}\| \leq \frac{1}{2}\alpha_t^{-\frac{1}{2}}. \quad (8)$$

Substituting the estimates (6) and (8) in inequality (5), we obtain

$$\|\hat{\vartheta}_t - \bar{\vartheta}\| \leq \|\vartheta^\circ - \bar{\vartheta}\| + h\left(\frac{\log(tr(Q_t))}{\alpha_t}\right)^{\frac{1}{2}},$$

h being a suitable constant.
Since by definition (4) $\alpha_t = (\log(tr(Q_t)))^{1+\delta}$ and from Assumption 2 $\lim_{t \to \infty} tr(Q_t) = \infty$, we then obtain that $\forall \epsilon > 0$ there exists a time instant τ such that $\|\hat{\vartheta}_t - \bar{\vartheta}\| \leq \|\vartheta^\circ - \bar{\vartheta}\| + \epsilon$, $\forall t \geq \tau$. By Assumption 3, this implies that there exists a finite time instant $\bar{t}$ such that $\hat{\vartheta}_t \in S(\bar{\vartheta}, r)$, $\forall t \geq \bar{t}$, *i.e.* point *ii)*. □

Part *ii)* in Theorem 1 shows that $\hat{\vartheta}_t \in S(\bar{\vartheta}, r)$, that is the projection operation (2.5) is disconnected, in the long run. As a consequence of this fact, the estimate $\bar{\vartheta}_t$ preserves closed-loop properties similar to those of the least squares algorithm. In addition, the uniform controllability of the model is guaranteed. This is precisely stated in Theorem 2 below. Preliminarily, we remind that a standard measure of the controllability of model $y_t = \varphi_{t-1}^T \vartheta + n_t$ is given by the absolute value of the determinant of the Sylvester matrix, namely $Sylv(\vartheta)$ (see *e.g.* [9]).

Theorem 2 *(Properties of the estimate $\bar{\vartheta}_t$)*
i) There exists a constant $c > 0$ such that $|det(Sylv(\bar{\vartheta}_t))| \geq c$, $\forall \, t$, a.s..

ii) Assume that u_t is $\mathcal{F}_t$-measurable. Then, the identification error satisfies a.s. the following bound

$$\|\vartheta^\circ - \bar{\vartheta}_t\|^2 = O\left(\frac{(\log(\lambda_{max}(\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q)))^{1+\delta}}{\lambda_{min}(\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q)}\right).$$

Proof.
Part i) Since the absolute value of the Sylvester matrix determinant is a continuous function of the system parameter ϑ and it is strictly positive for any $\vartheta \in S(\bar{\vartheta}, r)$, we can take $c := \min_{\vartheta \in S(\bar{\vartheta}, r)} |det(Sylv(\vartheta))| > 0$. Point *i)* then immediately follows from the definition of $\bar{\vartheta}_t$ in equation (2.5).

Part ii) Let us rewrite the performance index $D_t(\vartheta)$ as a function of the least squares estimate $\hat{\vartheta}_t^{LS} = arg \min_{\vartheta \in R^{n+m+1}} V_t(\vartheta)$:

$$D_t(\vartheta) = (\vartheta - \hat{\vartheta}_t^{LS})^T [\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q](\vartheta - \hat{\vartheta}_t^{LS})$$
$$+ \alpha_t \|\vartheta - \bar{\vartheta}\|^2 + V_t(\hat{\vartheta}_t^{LS}).$$

From the definition of $\hat{\vartheta}_t$, it follows that

$$D_t(\hat{\vartheta}_t) - V_t(\hat{\vartheta}_t^{LS}) \leq D_t(\vartheta^\circ) - V_t(\hat{\vartheta}_t^{LS}) = O(\alpha_t) \quad (9)$$

a.s., where the last equality is a consequence of the already cited Theorem 1 in [8] and of the boundedness of ϑ°. Consider now the inequality

$$\begin{aligned}
&(\vartheta^\circ - \hat{\vartheta}_t)^T[\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q](\vartheta^\circ - \hat{\vartheta}_t) \\
&\leq 2\{(\vartheta^\circ - \hat{\vartheta}_t^{LS})^T[\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q](\vartheta^\circ - \hat{\vartheta}_t^{LS}) \\
&+(\hat{\vartheta}_t^{LS} - \hat{\vartheta}_t)^T[\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q](\hat{\vartheta}_t^{LS} - \hat{\vartheta}_t)\}.
\end{aligned}$$

Since in view of equation (9) both terms in the right-hand-side are almost surely $O(\alpha_t)$, we get

$$||\vartheta^\circ - \hat{\vartheta}_t||^2 = O\left(\frac{\alpha_t}{\lambda_{min}(\sum_{k=1}^{t} \varphi_{k-1}\varphi_{k-1}^T + Q)}\right) \quad \text{a.s..}$$

Since $\bar{\vartheta}_t = \hat{\vartheta}_t$, $\forall t \geq \bar{t}$ (point *ii)* in Theorem 1) and also recalling definition (4) of α_t, point *ii)* immediately follows. □

4 Stabilization via adaptive pole placement

Let $A^*(q^{-1}) = \sum_{i=0}^{2s-1} a_i^* q^{-i}$ ($s = \max\{n, m+d\}$) be an arbitrary stable polynomial with $a_0^* = 1$. Given a controllable system $y_t = \varphi_{t-1}^T \vartheta + n_t$, it is known (see *e.g.* [10]) that there exist unique polynomials $L(\vartheta; q^{-1}) = \sum_{i=0}^{s-1} l_i(\vartheta) q^{-i}$ and $R(\vartheta; q^{-1}) = \sum_{i=1}^{s-1} r_i(\vartheta) q^{-i}$ such that the closed-loop system

$$\begin{cases} y_t = [1 - A(\vartheta; q^{-1})]\, y_t + B(\vartheta; q^{-1})\, u_t + n_t \\ u_t = L(\vartheta; q^{-1})\, (y_t - y_t^*) + R(\vartheta; q^{-1})\, u_t \end{cases}$$

has characteristic polynomial $A^*(q^{-1})$, $\{y_t^*\}$ being a bounded and deterministic reference signal.
The coefficients $\{l_i(\vartheta)\}_{i=0,\dots,s-1}$ and $\{r_i(\vartheta)\}_{i=1,\dots,s-1}$ are in fact given by the following equation

$$\begin{bmatrix} 1 \\ -r_1 \\ \vdots \\ -r_{s-1} \\ -l_0 \\ \vdots \\ -l_{s-1} \end{bmatrix} = Sylv(\vartheta)^{-1} \begin{bmatrix} 1 \\ a_1^* \\ a_2^* \\ \vdots \\ a_{s-1}^* \\ \vdots \\ a_{2s-1}^* \end{bmatrix}. \quad (10)$$

In adaptive control the system parameter vector ϑ° is unknown. Then, according to the certainty equivalence principle, one chooses the control law so as to assign the closed-loop poles to the estimated system, as if it were the true system. If the estimated system accurately describes the true system, one should be succeeding in stabilizing the true system through this procedure. When dealing with time-varying estimated parameters, it is a wise and widely adopted - see *e.g.* [11] - strategy to update the estimate at a slower rate than the system variables updating rate. In this way, the slow time variability of the corresponding adaptive control law cannot hamper the overall stability of the closed-loop control system. Following this idea, we incorporate a freezing feature in the estimator. Precisely, we use the following parameter estimate

$$\vartheta_t = \begin{cases} \bar{\vartheta}_t, & \text{if } t = iT,\, i = 0, 1, 2, \dots \\ \vartheta_{t-1}, & \text{otherwise.} \end{cases} \quad (11)$$

In Theorem 3 it is shown that the updating time interval T can be selected so as to uniformly stabilize the estimated time-varying system. This result is fundamental in order to prove that the pole placement control law tuned to the estimated parameter with freezing (11) is in fact able to stabilize the unknown true system (Theorem 4).
We first introduce some notations to which we shall refer in the theorem below.
Consider the autonomous estimated system:

$$\begin{cases} y_t = [1 - A(\vartheta_t; q^{-1})]\, y_t + B(\vartheta_t; q^{-1})\, u_t \\ u_t = L(\vartheta_t; q^{-1})\, y_t + R(\vartheta_t; q^{-1})\, u_t \end{cases}. \quad (12)$$

By letting $x_t := [y_t \dots y_{t-p+1}\, u_t \dots u_{t-q+1}]^T$ with $p = \max\{s-1, n\}$, $q = \max\{s-1, d+m\}$, system (12) can be given the state space representation

$$x_t = F(\vartheta_t)\, x_{t-1},$$

where matrix $F(\vartheta)$ is defined in equation (13) below and $a_i = 0$ if $i > n$, $l_i(\vartheta) = 0$ if $i > s-1$, $b_i = 0$ if $i < d$ or $i > d+m$, $r_i(\vartheta) = 0$ if $i > s-1$.

Theorem 3
Fix a constant $\mu < 1$ (*contraction* constant) and set

$$T(\vartheta) := \inf\{\tau \in Z_+ : ||F(\vartheta)^\tau|| \leq \mu\}.$$

Then, $\sup_{\vartheta \in S(\bar{\vartheta}, r)} T(\vartheta)$ is finite and with the position $T := \sup_{\vartheta \in S(\bar{\vartheta}, r)} T(\vartheta)$ in (11) we have that the autonomous system $x_t = F(\vartheta_t)\, x_{t-1}$ is a.s. exponentially stable, uniformly in time: $||x_t|| \leq M \bar{\nu}^{t-t^*} ||x_{t^*}||$, $\forall t$, t^*, $t^* \leq t$, where $M > 0$ and $0 < \bar{\nu} < 1$ are suitable constants.

Proof.
The proof of the theorem is omitted due to space limitations. However, based on the stability of the *frozen* matrix $F(\vartheta_t)$ and its slow time-variability (see (11)), the interested reader can work out by himself the rather simple proof. □

$$F(\vartheta)=\begin{bmatrix} a_1 & \dots & a_p & b_1 & \dots & b_q \\ 1 & & & & & \\ & \ddots & & & \vdots & \\ & & 0 & & & \\ l_0(\vartheta)a_1+l_1(\vartheta) & \dots & l_0(\vartheta)a_p+l_p(\vartheta) & l_0(\vartheta)b_1+r_1(\vartheta) & \dots & l_0(\vartheta)b_q+r_q(\vartheta) \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \tag{13}$$

Theorem 4 (*L^2-stability*)
With the same positions as in Theorem 3, the closed-loop system

$$\begin{cases} y_t = [1 - A(\vartheta^\circ; q^{-1})]\, y_t + B(\vartheta^\circ; q^{-1})\, u_t + n_t \\ u_t = L(\vartheta_t; q^{-1})\,(y_t - y_t^*) + R(\vartheta_t; q^{-1})\, u_t \end{cases} \tag{14}$$

is pathwise L^2-stable: $\limsup_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}[y_t^2+u_t^2] < \infty$ a.s.. □

The proof of Theorem 4 is based on the following Lemma whose technical proof is omitted due to space limitations.

Lemma 1
Consider a sequence of l-dimensional vectors $\{v_t\}$ such that the following assumptions are satisfied:

i) $\{v_t\}$ is bounded: $\|v_t\| \leq \bar{v}$, $\forall t$;

ii) $\{v_t\}$ is piecewise constant: $v_t = v_{t_i}$, $t \in [t_i, t_{i+1})$, where t_i is such that $T := \sup_i (t_{i+1} - t_i) < \infty$.

Given a second l-dimensional vector sequence $\{z_t\}$ such that

iii) $\sum_{t=0}^{t_i} (z_t^T v_{t_i})^2 = o(\sum_{t=0}^{t_i} \|z_t\|^2) + O(1)$, $\forall i$,

it follows that

$$\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} (z_t^T v_t)^2 = o(\sum_{t=0}^{N}\|z_t\|^2 + N),$$

where $\mathcal{B}_N$ is a set of instant points which depends on N, whose cardinality, however, is upper bounded by Tl for any N: $|\mathcal{B}_N| \leq Tl$, $\forall N$. □

Proof of Theorem 4.
Fix a time instant point $N > 0$.
Since $\{\vartheta^\circ - \vartheta_t\}$ is bounded and constant over $[iT, (i+1)T)$ (see (11)), $\forall\ i$, $T < \infty$ (see Theorem 3), and $\sum_{t=0}^{iT} (\varphi_{t-1}^T(\vartheta^\circ - \vartheta_{iT}))^2 = o(\sum_{t=0}^{iT}\|\varphi_{t-1}\|^2) + O(1)$, $\forall\, i$ (see Theorem 2), we can apply Lemma 1 to get an upper bound on the pathwise square average of the identification error $e_t := \varphi_{t-1}^T(\vartheta^\circ - \vartheta_t)$ up to time N. Such a bound is given by

$$\frac{1}{N}\sum_{t=0,\ t\notin\mathcal{B}_N}^{N} e_t^2 = \frac{1}{N} o(\sum_{t=0}^{N}\|\varphi_{t-1}\|^2 + N), \tag{15}$$

where the set $\mathcal{B}_N$ of instant points depends on N, but has a cardinality which is upper bounded by $(n+m+1)N$ for any N.
In the time interval $[0, N]$ the state vector $x_t = [y_t\, y_{t-1} \dots y_{t-p+1}\, u_t\, u_{t-1} \dots u_{t-q+1}]^T$ associated with system (14) is governed by the following equation

$$x_t = \begin{cases} F(\vartheta_t)\, x_{t-1} + G(\vartheta_t)[e_t + n_t] - HL(\vartheta_t, q^{-1})y_t^*, & t \notin \mathcal{B}_N \\ F^\circ(\vartheta_t)\, x_{t-1} + G(\vartheta_t) n_t - HL(\vartheta_t, q^{-1})y_t^*, & t \in \mathcal{B}_N \end{cases}, \tag{16}$$

where $F^\circ(\vartheta)$ is given in equation (17) below and vectors $G(\vartheta)$ and H are respectively given by $G(\vartheta) = [1\,0\dots0\,l_0(\vartheta)\,0\dots0]^T$ and $H = [0\,0\dots0\,1\,0\dots0]^T$.
Since ϑ_t belongs to the compact set $S(\bar{\vartheta}, r)$ and $F^\circ(\vartheta)$ and $G(\vartheta)$ are continuous function of ϑ, $\vartheta \in S(\bar{\vartheta}, r)$, we then have that $\|F^\circ(\vartheta_t)\| \leq h$ and $\|G(\vartheta_t)\| \leq h$, h being a suitable constant. From this fact and the uniform exponential stability of $x_t = F(\vartheta_t)x_{t-1}$ (Theorem 3), it is easily understood that the state vector x_t generated by system (16) can be bounded as follows

$$\|x_t\| \leq (h\,M)^{|\mathcal{B}_N|}\{\bar{\nu}^{t-|\mathcal{B}_N|}\|x_0\| + \sum_{k=0}^{t} \bar{\nu}^{t-k-|\mathcal{B}_N|} [|n_k| + |\bar{y}_k^*|] + \sum_{k=0,k\notin\mathcal{B}_N}^{t} \bar{\nu}^{t-k-|\mathcal{B}_N|}|e_k|\},\ \ t \leq N,$$

where $\bar{y}_k^* := L(\vartheta_k, q^{-1})y_k^*$. As a consequence, we also have

$$\|x_t\|^2 \leq k_1\{\bar{\nu}^{2t}\|x_0\|^2 + \sum_{k=0}^{t} \bar{\nu}^{t-k}[n_k^2 + (\bar{y}_k^*)^2] + \sum_{k=0,k\notin\mathcal{B}_N}^{t} \bar{\nu}^{t-k} e_k^2\},\ \ t \leq N,$$

k_1 being a suitable constant, independent of N.
Bearing in mind the definition of the observation vector

$$F^\circ(\vartheta) = \begin{bmatrix} a_1^\circ & \cdots & a_p^\circ & b_1^\circ & \cdots & b_q^\circ \\ 1 & & & & & \\ & \ddots & & \vdots & & \\ & & 0 & & & \\ l_0(\vartheta)a_1^\circ + l_1(\vartheta) & \cdots & l_0(\vartheta)a_p^\circ + l_p(\vartheta) & l_0(\vartheta)b_1^\circ + r_1(\vartheta) & \cdots & l_0(\vartheta)b_q^\circ + r_q(\vartheta) \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \tag{17}$$

φ_t, from this last inequality we get

$$\frac{1}{N}\sum_{t=0}^{N}\|\varphi_t\|^2 \leq k_2\{\frac{1}{N}\|x_0\|^2 + \frac{1}{N}\sum_{t=0}^{N} n_t^2 + \frac{1}{N}\sum_{t=0}^{N}(\bar{y}_t^*)^2 + \frac{1}{N}\sum_{t=0, t\notin \mathcal{B}_N}^{N} e_t^2\} \tag{18}$$

where k_2 is a suitable constant.

The first term in the right-hand-side of this expression vanishes as N tends to infinity. As for the second term $\frac{1}{N}\sum_{t=0}^{N} n_t^2$, by exploiting Chow's theorem (see *e.g.* [12]) and Assumption 1 on the noise, it can be easily shown that it is almost surely bounded. The third term is bounded as well since $\bar{y}_t^*$ is bounded. $\bar{y}_t^*$ is in fact given by a linear combination of s samples of the reference signal y_t^* weighted with the coefficients $\{l_i(\vartheta)\}_{i=0,\ldots,s-1}$, which are continuous and therefore bounded functions of ϑ, $\vartheta \in S(\bar{\vartheta}, r)$ (see (10)).

By using these estimates of the first three terms in the right-hand-side of inequality (18) and applying equality (15), we obtain

$$\frac{1}{N}\sum_{t=0}^{N}\|\varphi_t\|^2 = O(1) + o(\frac{1}{N}\sum_{t=0}^{N}\|\varphi_{t-1}\|^2),$$

which implies that $\frac{1}{N}\sum_{t=0}^{N}\|\varphi_t\|^2$ remains bounded. Then, the thesis immediately follows. □

5 Conclusions

In the present contribution, we have introduced a new identification algorithm securing the estimated system controllability, which is widely recognized as a central problem in adaptive control. The proposed approach requires some a-priori knowledge on the region to which the true parameter belongs, but, in contrast with other methods, it has the advantage to be easily implementable. It is therefore suggested as an effective solution to the controllability problem in all the situations in which the required a-priori knowledge is available.

References

[1] G.C. Goodwin and K.S. Sin. "Adaptive control of nonminimum phase systems." *IEEE Trans. on Automatic Control*, AC-26, 1981, pp. 478-483,.

[2] Ph. de Larminat. "On the stabilizability condition in indirect adaptive control." *Automatica*, 20, 1984, pp. 793-795.

[3] R. Lozano and X.H. Zhao. "Adaptive pole placement without excitation probing signals." *IEEE Trans. on Automatic Control*, AC-39, 1994, pp. 47-58.

[4] M.C. Campi. "The problem of pole-zero cancellation in transfer function identification and application to adaptive stabilization." *Automatica*, 32, 1996, pp. 849-857.

[5] G. Kreisselmeier. "A robust indirect adaptive control approach." *Int. J. Contr.*, 43, 1986, pp. 161-175.

[6] K.A. Ossman and E.D. Kamen. "Adaptive regulation of MIMO linear discrete-time systems without requiring a persistent excitation." *IEEE Trans. on Automatic Control*, AC-32, 1987, pp. 397-404,.

[7] R.H. Middleton, G.C. Goodwin, D.J. Hill, and D.Q. Mayne. "Design issues in adaptive control." *IEEE Trans. on Automatic Control*, AC-33, 1988, pp. 50-58.

[8] T.L. Lai and C.Z. Wei. "Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems." *Ann. Statist.*, 10, 1982, pp. 154-166.

[9] T. Kailath. *Linear Systems Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1980.

[10] G.C. Goodwin and K.S. Sin. *Adaptive Filtering, Prediction and Control.* Englewood Cliffs, NJ: Prentice-Hall, 1984.

[11] R. Ortega and Y. Tang. "Robustness of adaptive controllers - A survey." *Automatica*, 25, 1989, pp. 651-677.

[12] H.F. Chen and L. Guo. *Identification and Stochastic Adaptive Control.* Birkhäuser, Boston, 1991.