Decision making in an uncertain environment: the scenario-based optimization approach

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A general fact:

- convex optimization is easy

but

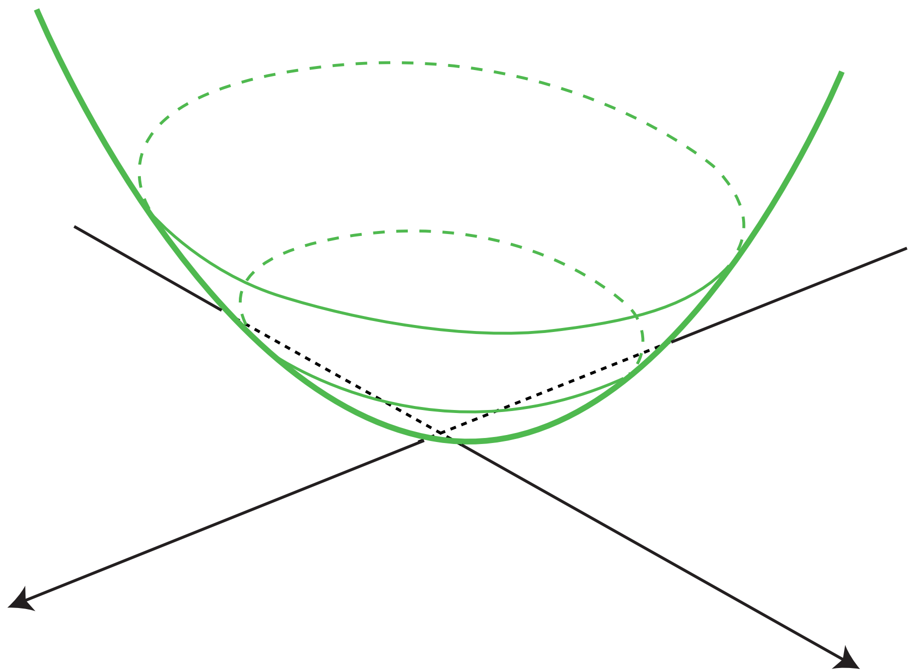
- robust convex optimization is hard

$$\begin{aligned}&\min\ c^T x\\&\text{subject to: } f(x,\delta) \leq 0, \quad \forall \delta \in \Delta\end{aligned}$$

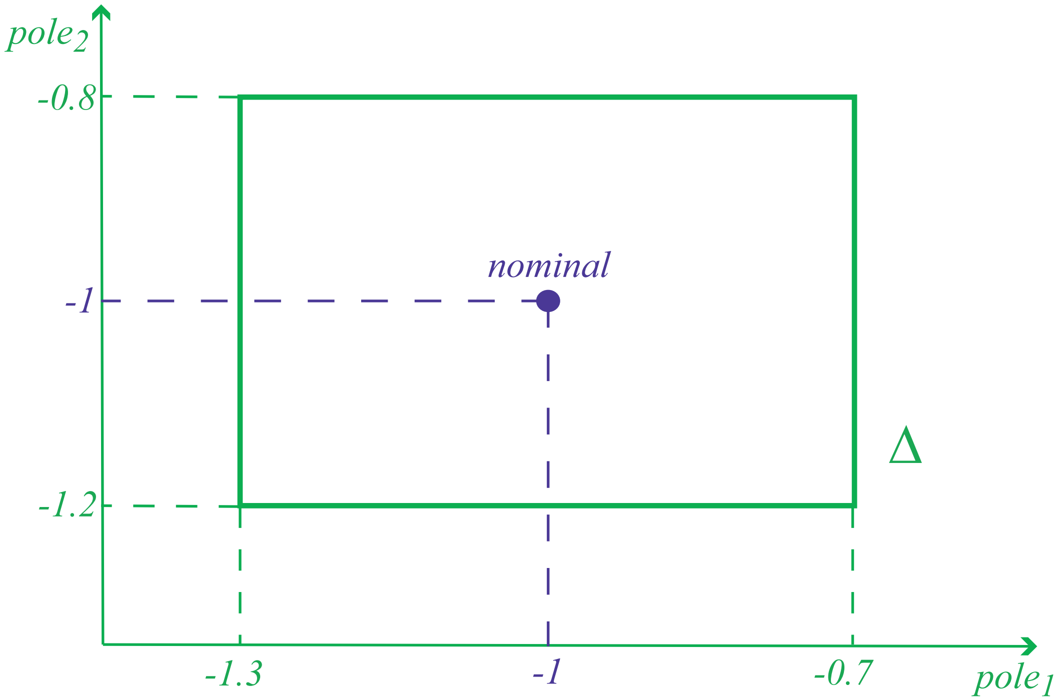
Example (stability)

$$\dot{x} = Ax$$

$$\begin{cases} P \succ 0 \\ A^T P + PA \prec 0 \end{cases} \qquad LMI - convex$$



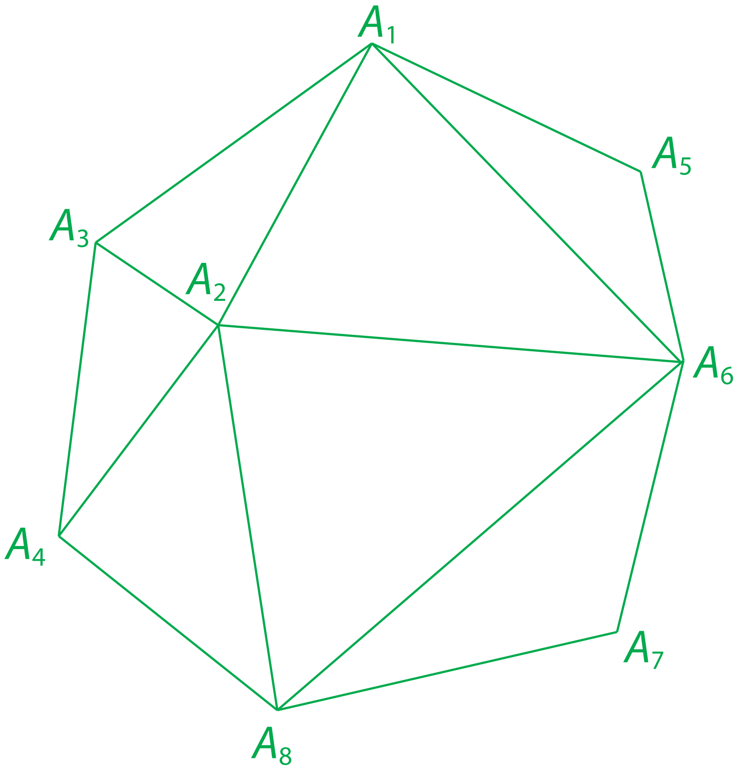
Uncertainty - robust



$$\dot{x} = A(\delta)x$$

$$\begin{cases} P \succ 0 \\ A(\delta)^T P + PA(\delta) \prec 0 \qquad \forall \delta \in \Delta \end{cases}$$

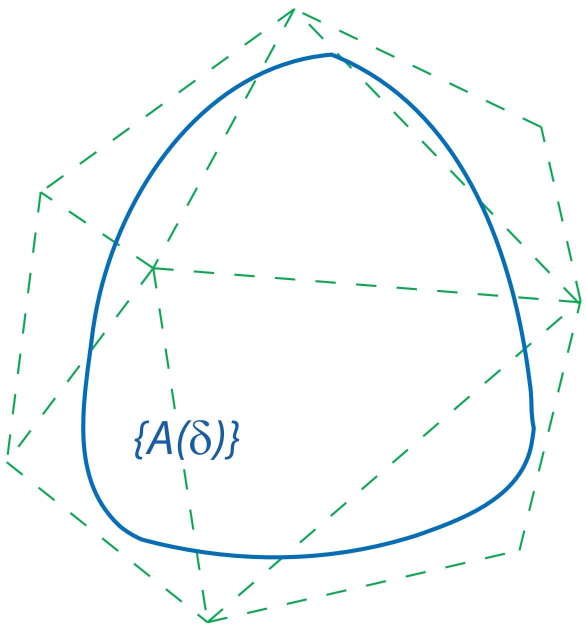
infinite number of constraints!!!



$$A(\delta) = \textstyle\sum_i \alpha_i A_i$$
$$(\text{convex: } 0 \leq \alpha_i \leq 1 \ \ \textstyle\sum_i \alpha_i = 1)$$

$$\begin{cases} P \succ 0 \\ A_1^T P + P A_1 \prec 0 \\ \vdots \\ A_n^T P + P A_n \prec 0 \end{cases}$$

Towards generality



relaxation

$$\begin{cases} P \succ 0 \\ A(\delta)^T P + P A(\delta) \prec 0 \end{cases}$$

QS - Quadratic Stability

$-P = P_0 + \delta_1 P_1 + \cdots + \delta_m P_m$ AQS - Affine Quadratic Stability

$-P(z, \delta)$ *linear in z* GQS - Generalized Quadratic Stability

$-P(\delta)$ general case

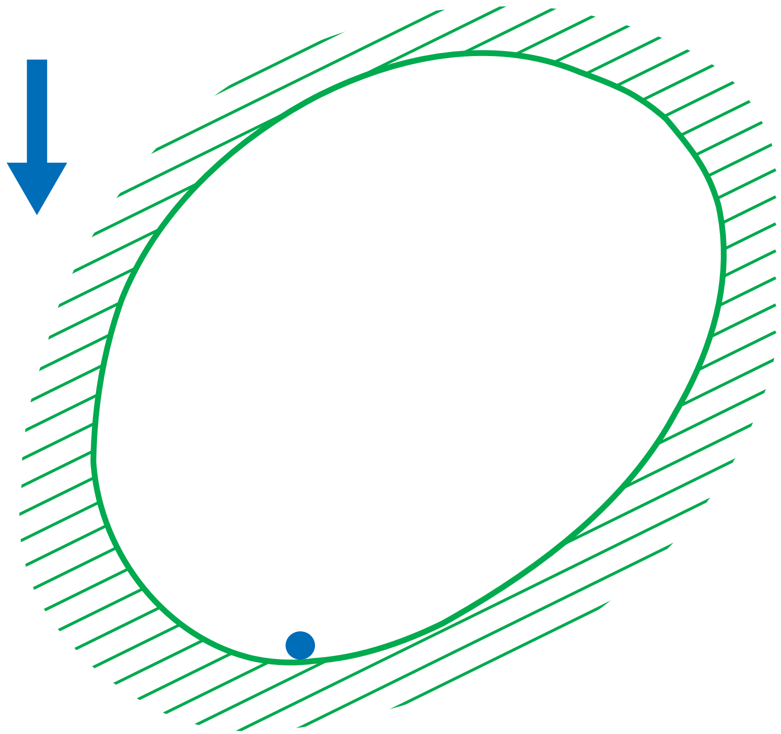
Value-at-Risk problem

$\ell(\xi, \delta) =$ (positive) loss of a portfolio of risky assets

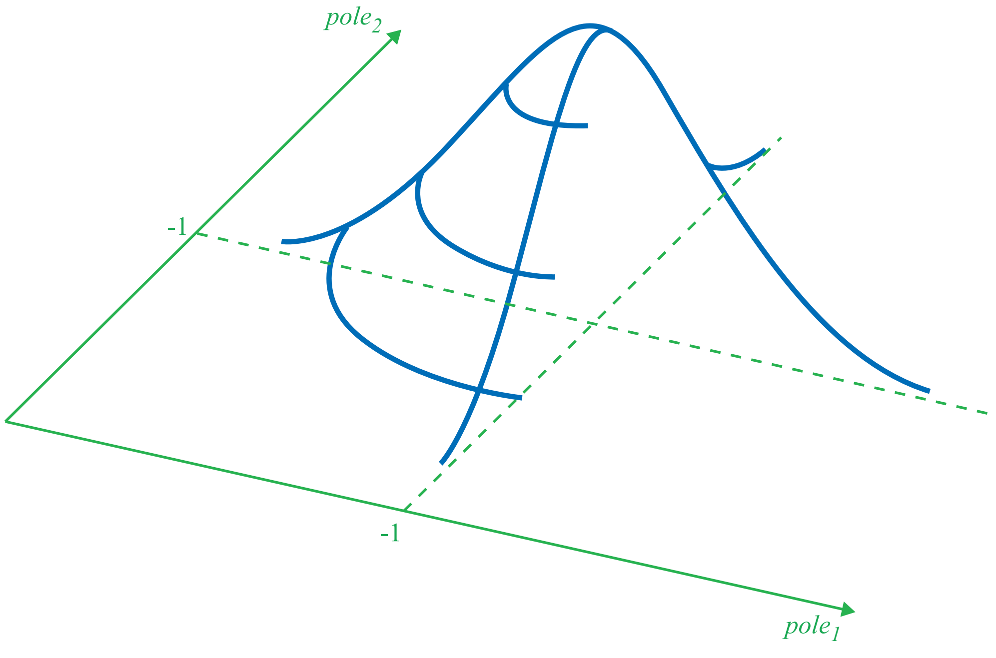
$\min_{\gamma,\xi} \ \gamma$
subject to: $\ell(\xi, \delta) \leq \gamma, \quad \forall \delta \in \Delta$

Robust Convex Optimization

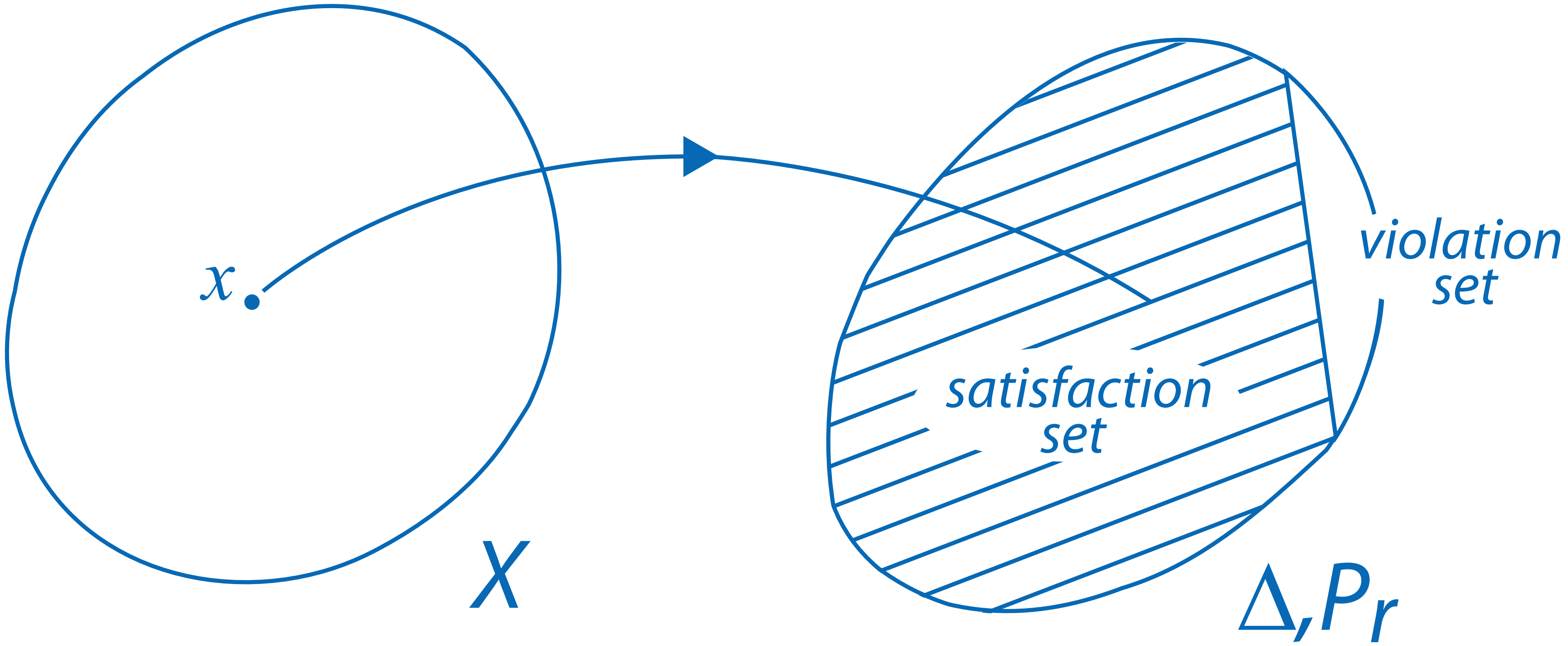
$$\min \; c^T x$$
$$\text{subject to: } f(x,\delta) \le 0, \quad \forall \delta \in \Delta$$



Uncertainty



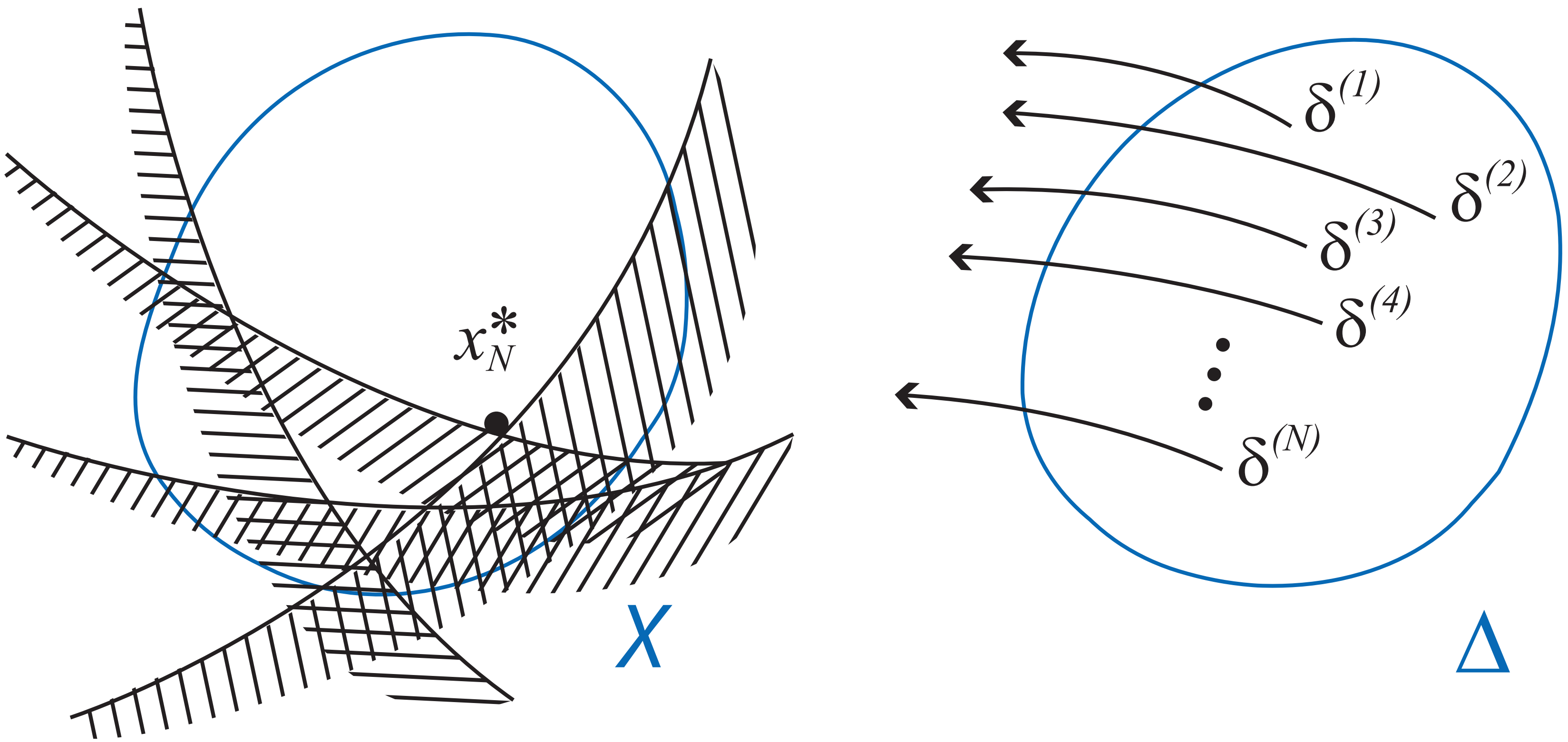
Violation set



$Pr(\text{violation set}) \leq \epsilon$

- chance-constrained optimization

The "Scenario" Paradigm



SCP_N = scenario convex program

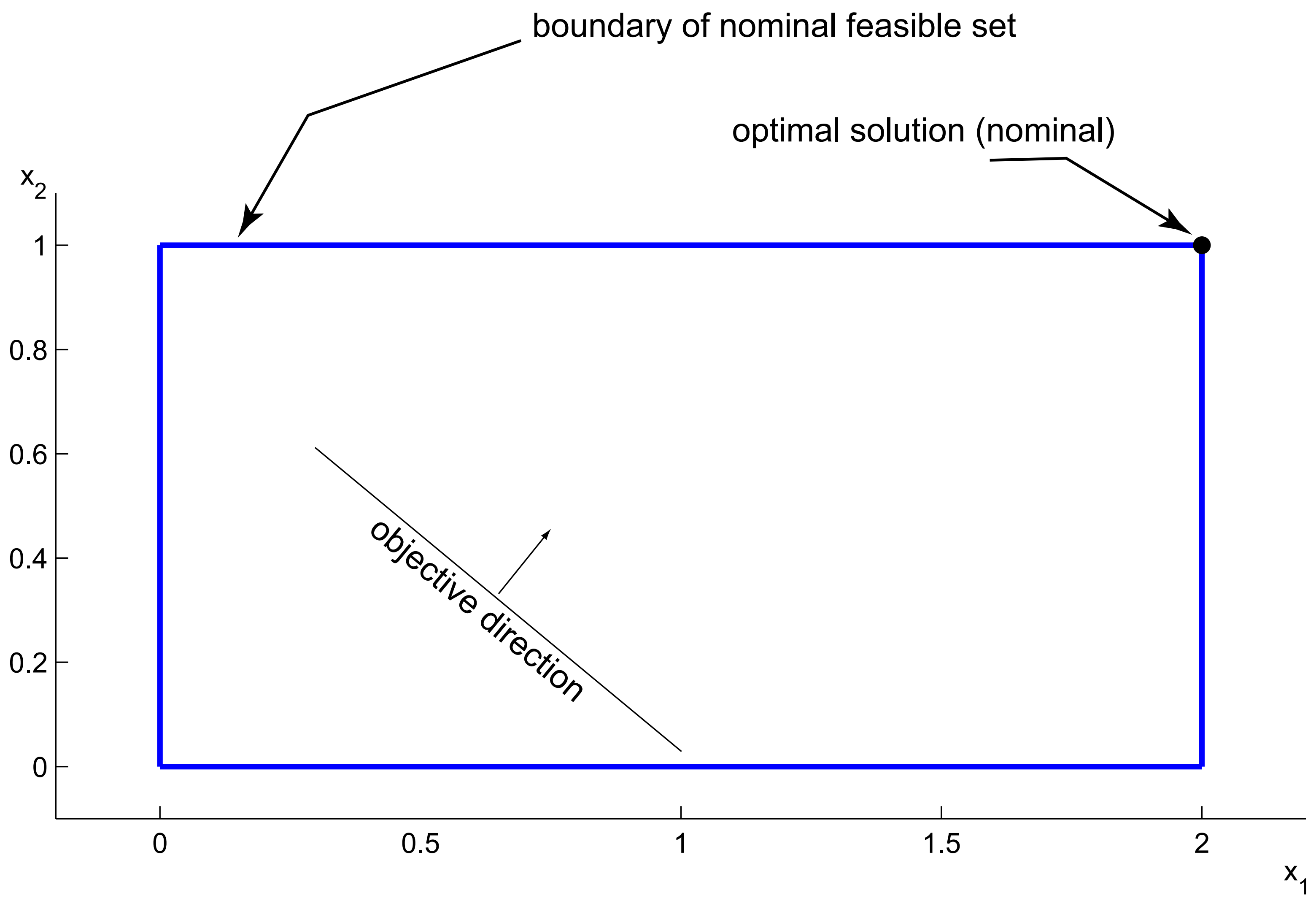
- SCP_N is a standard finite convex optimization problem
- x_N^* is superoptimal

Fundamental question:

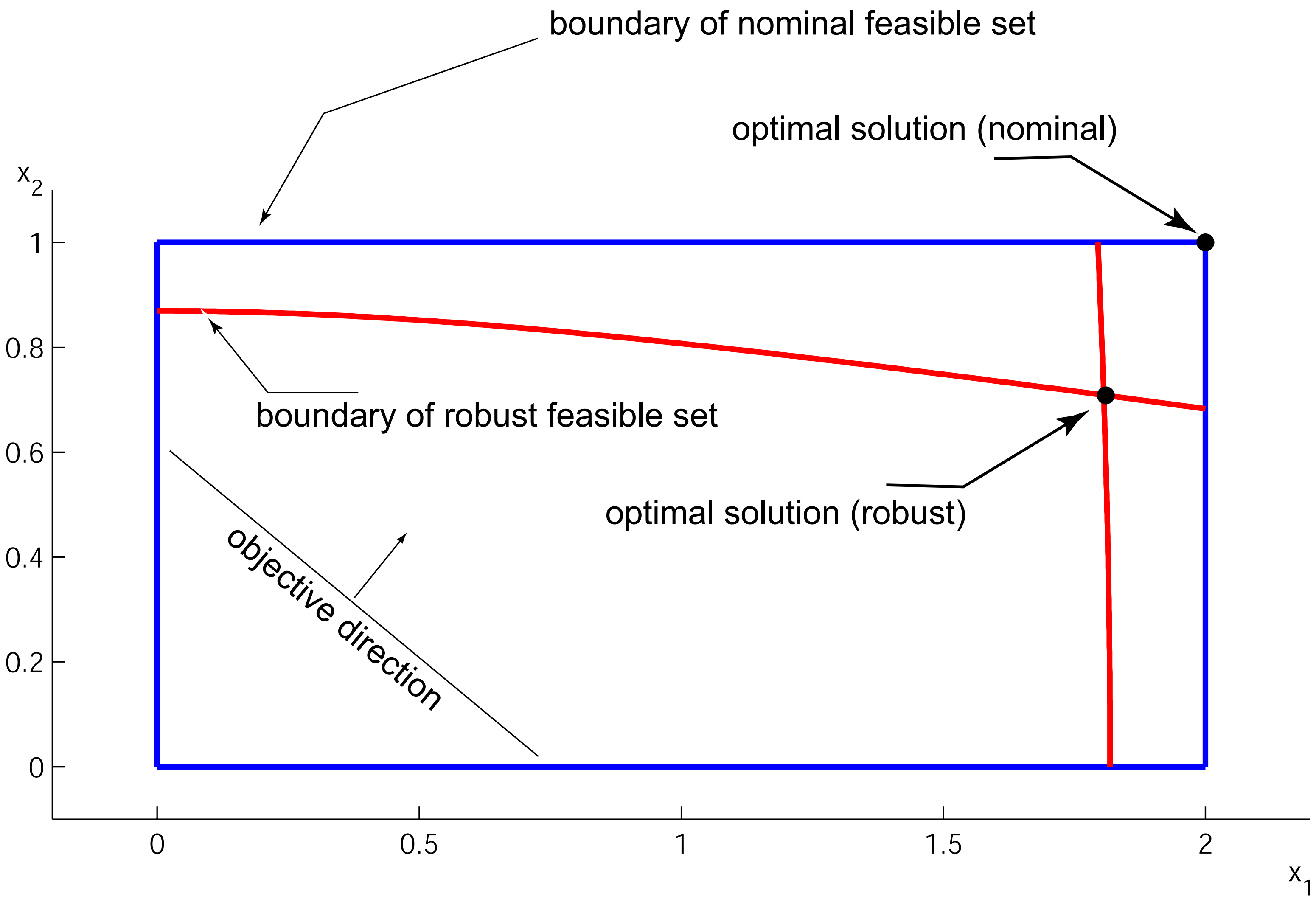
$\boxed{\text{how feasible is } x_N^*?}$

Example

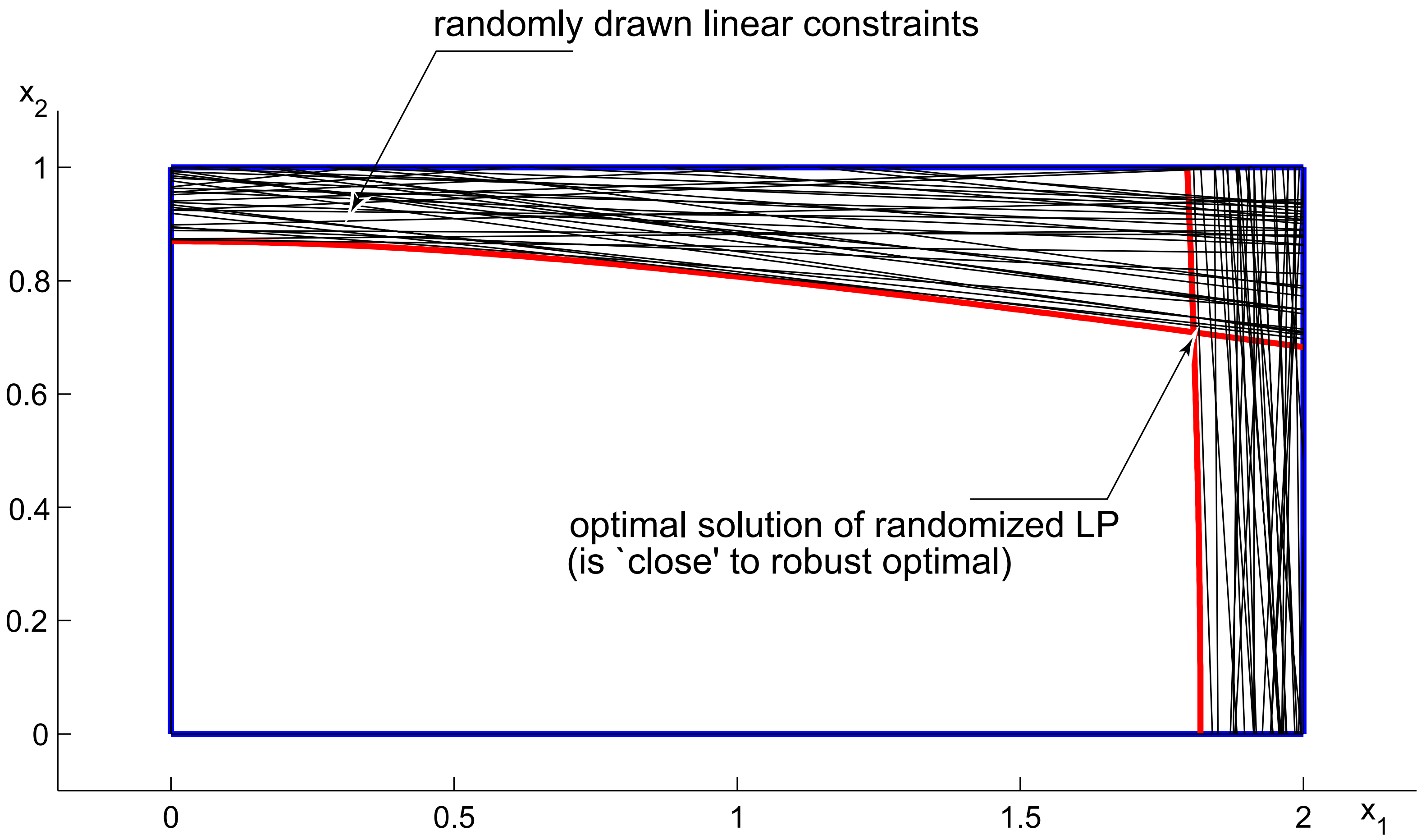
$$
\begin{array}{lll}
\text{RLP:} & \min & c^T x \\
& \text{subject to} & a_1^T x \le 2, \quad a_1^T = [1 \;\; 0] + \rho_1 \delta_1, \quad \rho_1 = 0.1, \; |\delta_1| \le 1 \\
& & a_2^T x \le 1, \quad a_2^T = [1 \;\; 0] + \rho_2 \delta_2, \quad \rho_2 = 0.15, \; |\delta_2| \le 1 \\
& & a_3^T x \le 0, \quad a_3^T = [-1 \;\; 0] \\
& & a_4^T x \le 0, \quad a_4^T = [0 \;\; -1]
\end{array}
$$



boundary of nominal feasible set
optimal solution (nominal)
x2
1
0.8
0.6
0.4
0.2
0
objective direction
0
0.5
1
1.5
2
x1



boundary of nominal feasible set
optimal solution (nominal)
boundary of robust feasible set
optimal solution (robust)
objective direction
x_2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x_1


randomly drawn linear constraints
optimal solution of randomized LP
(is `close' to robust optimal)
x2
x1
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2

Fundamental question:

$\boxed{\text{how feasible is } x_N^*?}$

generalization $\Longrightarrow$ need for structure

Good news: the structure we need is convexity

- double role of convexity:
  - practice (computation)
  - theory (generalization)

Theorem

Fix $\epsilon \in (0,1)$ (violation parameter)

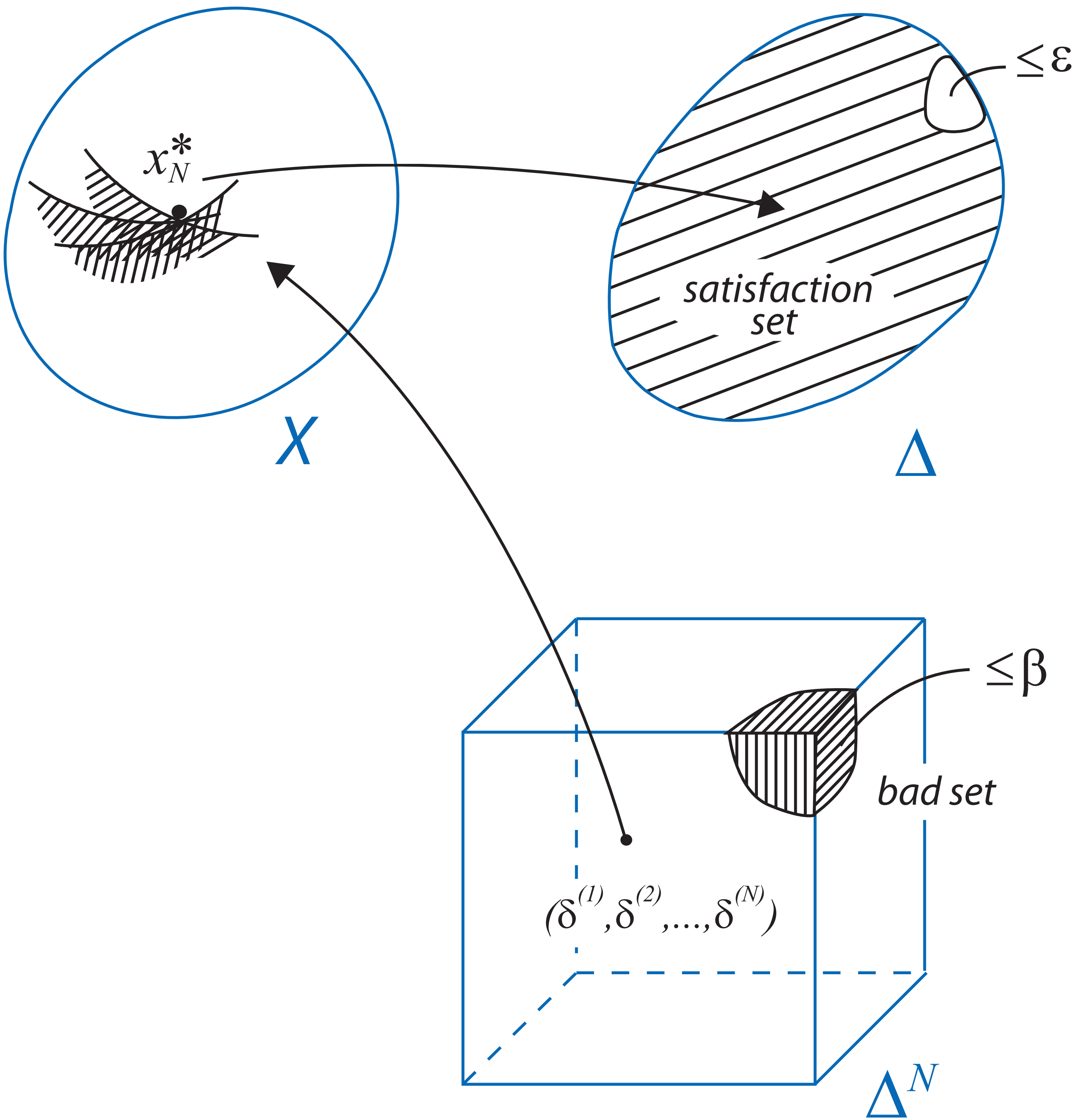
$\beta \in (0,1)$ (confidence parameter)

If $N \geq N(\epsilon,\beta) \doteq \frac{2}{\epsilon}\ln\frac{1}{\beta} + 2n_x + \frac{2n_x}{\epsilon}\ln\frac{2}{\epsilon}$,

then,

with probability $\geq 1-\beta$,

x_N^* is ϵ-level robustly feasible.



x_N^*
$\leq \varepsilon$
satisfaction set
X
Δ
$\leq \beta$
bad set
$(\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)})$
Δ^N

Extensions:

- SCP_N is unfeasible
- x_N^* is not unique
- SCP_N is feasible, but x_N^* does not exist

Comments:

$N \geq \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n_x + \frac{2n_x}{\epsilon} \ln \frac{2}{\epsilon}$

- N usually tractable by standard solvers
- N easy to compute
- N independent of Pr
- permits to address problems otherwhise intractable

$$\underline{Ex}: \text{ stability of } A(\delta)$$
$$P(x, \delta) \quad GQS$$

- even when RCP is tractable, SCP_N gives a way to trade probability of violation for performance

$\rightarrow \epsilon =$ tuning knob

Multi-participant decision making

Value-at-Risk problem:

$$
\begin{aligned}
&\min_{\gamma,\xi} \;\; \gamma \\
&\text{subject to: } Pr_1\{\ell(\xi,\delta_1) > \gamma\} \le \epsilon \\
&\phantom{\text{subject to: }} Pr_2\{\ell(\xi,\delta_2) > \gamma\} \le \epsilon
\end{aligned}
$$

Theorem

$\epsilon, \beta \longrightarrow N$ constraints $\delta_1 + N$ constraints δ_2.

Then, with probability $\ge 1-\beta$, $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2 \le \epsilon$.

If e.g. $\epsilon_1 = \epsilon$, then $\epsilon_2 = 0$

Example (stability-synthesis)

$$\dot{x} = \begin{bmatrix} 0.5\delta_2 & 1+\delta_1 \\ -(1+\delta_1)^2 & 2(0.1+0.5\delta_2)(1+\delta_1) \end{bmatrix} x + \begin{bmatrix} 10 \\ 15 \end{bmatrix} u$$

$|\delta_1| \leq 1, \quad |\delta_2| \leq 1$

Goal: design $u = Kx$ such that the closed-loop is quadratically stable

$$A_{cl}(\delta) = A(\delta) + BK$$

Lyapunov condition:

$$PA^T(\delta) + A(\delta)P + \underbrace{PK^T}_{=:Y^T} B^T + B \underbrace{KP}_{=:Y} \prec 0 \qquad \forall \delta \in \Delta$$

$$K = YP^{-1}$$

$$\min_{P,Y,\gamma} \quad \gamma$$

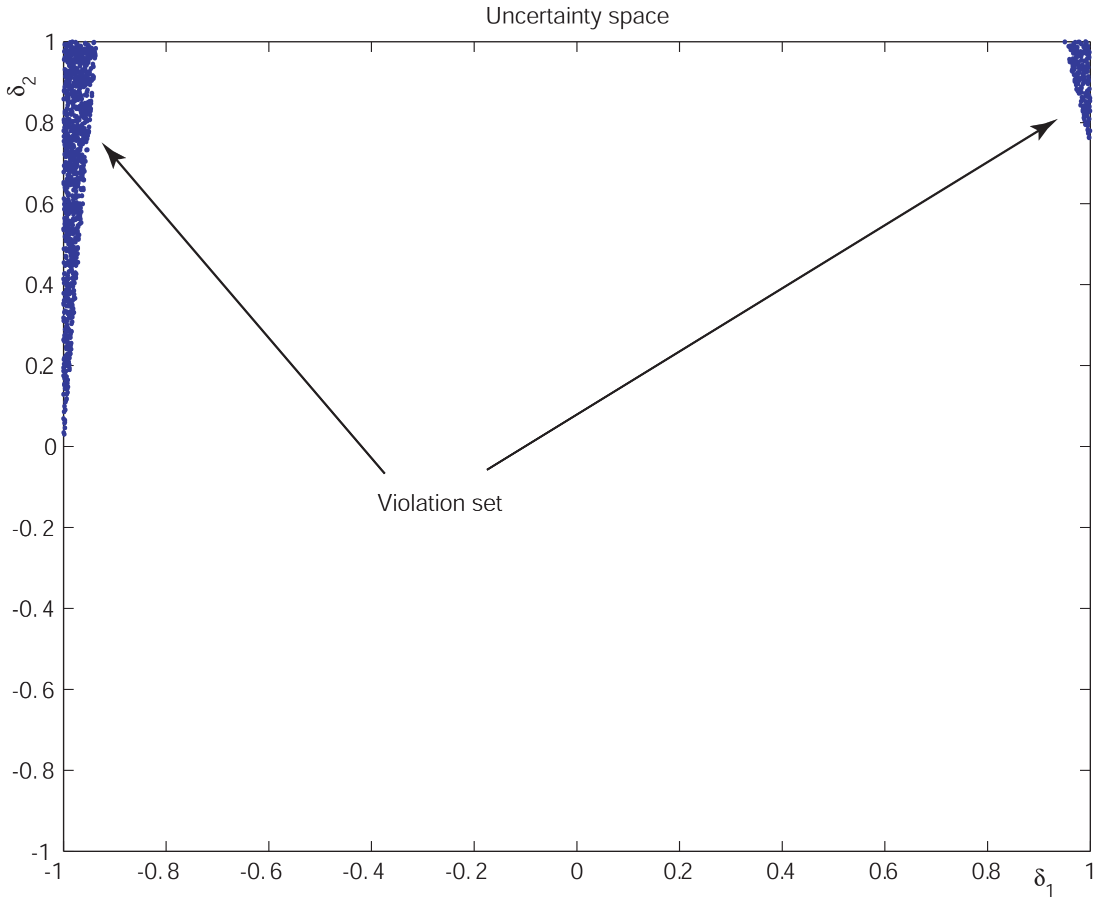
$$\text{subject to} \quad -I \preceq \begin{bmatrix} -P & 0 \\ 0 & PA^T(\delta) + A(\delta)P + Y^T B^T + BY \end{bmatrix} \preceq \gamma I$$

$$\left.\begin{array}{rcl} \epsilon & = & 0.05 \\ \beta & = & 0.001 \end{array}\right\} \rightarrow N = 1174$$

$$P = \left[\begin{array}{cc} 0.0273 & -0.0212 \\ -0.0212 & 0.4852 \end{array}\right], \quad Y = \left[\begin{array}{cc} -0.1620 & -0.2280 \end{array}\right]$$

$$K = [-6.5162 \quad -0.7550].$$

A-posteriori: Monte-Carlo analysis



$N = 100,000$

$\hat{\epsilon} = 0.0096$

Conclusions

- Finite convex optimization is simple, but semi-infinite convex optimization is hard in general

- The scenario approach offers a viable way to solve semi-infinite convex optimization problems in a risk-adjusted sense

- ϵ trades robustness for performance.
In certain problems (prediction) one may want to have $\epsilon = 0.2 - 0.3$

- The generalization result is valid for all convex problems