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Logic-based Switching for the Stabilization of Stochastic Systems in presence of Unmodeled Dynamics[1]

Maria Prandini and M.C. Campi
Dept. of Electronics for Automation
University of Brescia, Italy
{prandini,campi}@ing.unibs.it

Abstract

In this paper, we describe a supervisory control scheme for adaptively stabilizing an unknown discrete time linear system affected by a possibly unbounded noise. The scheme incorporates a switching logic mechanism, which, at adaptively selected event times, places in feedback with the system the controller designed for the model which is the best according to the least squares criterion. The event times are chosen so as to uniformly stabilize the estimated system.
We show that, when the controller selection is based on a reduced order model of the system and the unmodeled dynamics is sufficiently small, the introduced switching scheme is successful in stabilizing the system. Moreover, in absence of unmodeled dynamics, we are also able to characterize the switching scheme performance in terms of a self-tuning result.

1 Introduction

In this paper, we address the problem of controlling an unknown discrete time single-input/single-output system affected by a possibly unbounded stochastic noise.
In general, when the uncertainty on the system description is large, no single candidate controller in a given family is able to adequately regulate all the admissible models for the system. This motivates the use of a switching control scheme, where a supervisor decides on-line, based on the observations collected from the operating system, which is the best controller to be applied and when it is the case to switch to a different controller.
Logic-based switching controllers were first proposed in [1] and turned out to be a valid alternative to the more traditional continuously tuned adaptive controllers (see e.g. [2], [3]-[8]). Most of the stability results proven in the literature refer to continuous time systems. On the other hand, switching controllers are typically implemented digitally, hence studying switching control in a discrete time setting is useful to get a better insight into the actual behavior of a switching scheme.

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In the so-called "estimator-based" switching method [2], the controller selection is based on the system description in terms of a parameterized model. At each switching time instant, the supervisor chooses the controller that is designed for the model whose parameter minimizes a certain performance index. A commonly adopted performance index is some integral norm of the output estimation error. The reader is referred to [9] for alternative indices using the virtual reference concept.

After formulating the problem in Section 2, in Section 3 we describe the architecture of the proposed supervisory control scheme. Its main blocks are the *performance index generator* and the *switching logic*. The performance index generator computes the value of the performance index based on the input-output data collected from the system. We use the least squares cost which is shown to guarantee that the output estimation error is small compared to the signals involved in the loop, irrespectively of the excitation conditions. Moreover, this property remains valid when part of the system dynamics is neglected in the model, if the unmodeled dynamics is sufficiently small.
The switching logic generates a switching signal, which determines the candidate controller to be placed in closed-loop with the system. The controller selection is based on the value of the least squares performance index. The switching rate is slowed by making a *dwell time* elapse between consecutive switching times. Here, the dwell time is adaptively selected so as to guarantee the uniform exponential stability of the estimated system. This differentiates our contribution from [2, 4] where the dwell time is kept fixed.
In Section 4, we prove that the introduced switching control scheme is successful in stabilizing the system when the unmodeled dynamics is sufficiently small, and that this is ensured despite of the fact that the stochastic noise acting on the system is possibly unbounded. We then consider the ideal case when the system belongs to the model class. In this situation, if the system parameter were consistently estimated, then the switching controller will perfectly tune to the system under control, at least in the long run. Unfortunately, consistency cannot be guaranteed in general since it re-

quires appropriate excitation conditions, which are difficult to ensure in closed-loop. Yet, the performance of the switching control scheme can be characterized in terms of a self-tuning result.

2 Problem formulation

Our goal is to regulate a single-input/single-output discrete time system by means of a controller in a candidate controllers family. When the system to be controlled is not known and the uncertainty on its description is large, no single candidate controller would generally achieve an adequate performance when applied to each one of the admissible descriptions for the system. In order to deal with such a situation, we propose to adopt an adaptive control scheme where a supervisor orchestrates the switching among the candidate controllers, based on the observations collected on the system. The idea underlying the switching mechanism design is that, if the accrued information leads to an accurate description of the system, then the switching mechanism should select the candidate controller that is better tuned to the system.
Here, we consider controllers of the form

$$\mathcal{R}(\gamma; q^{-1}) u_t = \mathcal{S}(\gamma; q^{-1}) y_t, \qquad (1)$$

where the polynomials $\mathcal{R}(\gamma; q^{-1}) = 1 - \sum_{i=1}^{m_c} r_i q^{-i}$ and $\mathcal{S}(\gamma; q^{-1}) = \sum_{i=0}^{n_c} s_i q^{-i}$ in the unit-delay operator q^{-1} depend on $\gamma = [s_0 \ s_1 \dots s_{n_c} \ r_1 \ r_2 \dots r_{m_c}]^T \in \Gamma \subseteq \mathbb{R}^{n_c+m_c+1}$.
The controller selection is based on the following model for the system:

$$\mathcal{A}(\vartheta; q^{-1})\, y_{t+1} = \mathcal{B}(\vartheta; q^{-1})\, u_t + w_{t+1}, \qquad (2)$$

where signal w represents some white noise, and $\mathcal{A}(\vartheta; q^{-1}) = 1 - \sum_{i=1}^{n_s} a_i q^{-i}$ and $\mathcal{B}(\vartheta; q^{-1}) = \sum_{i=1}^{m_s} b_i q^{-i+1}$ are polynomials with parameter $\vartheta := [a_1 \ a_2 \dots a_{n_s} \ b_1 \ b_2 \dots b_{m_s}]^T$. Note that model (2) has the nice property to be linearly parameterized, since it can in fact be rewritten in the regression-like form:

$$y_{t+1} = \varphi_t^T \vartheta + w_{t+1}, \qquad (3)$$

where $\varphi_t := [\, y_t \ y_{t-1} \ \dots y_{t-n_s+1} \ u_t \ u_{t-1} \dots u_{t-m_s+1}\,]^T$.
We suppose that the model parameter ϑ belongs to a set Θ which can be either a finite or a compact subset of $\mathbb{R}^{n_s+m_s}$, and that $m_s \geq 1$ and $n_s \geq 1$, so that the regulation problem is well-posed and not trivially solved by setting $u_t = 0, t \geq 0$.
We require that the set of candidate controllers is sufficiently rich in that

Assumption 1 *There exists a map $\Sigma : \Theta \to \Gamma$ associating to each model (2) with parameter $\vartheta \in \Theta$ a controller (1) with parameter $\gamma = \Sigma(\vartheta) \in \Gamma$ that stabilizes it. In the case when Θ is a continuum of parameterization, we require Σ to be continuous over Θ.*

We denote by λ, $0 < \lambda < 1$, the *stability margin*, i.e., the maximum absolute value over Θ of the eigenvalues of the closed-loop system where the model with parameter ϑ is controlled by the controller with parameter $\Sigma(\vartheta)$. Note that λ is well-defined due to Assumption 1.

In Section 3, we propose an estimator-based switching scheme using the system description (2) and the controller mapping Σ. Then, in Section 4 we study its properties in the case when (2) is a reduced order model for the system that describes, for example, only that part of the system which includes its possibly unstable modes [10]. To be more precise, we consider the case when the true system is given by

$$y_{t+1} = \varphi_t^T \vartheta^\circ + v_t + w_{t+1},$$

where $\vartheta^\circ \in \Theta$ and v is the unmodeled dynamics signal. v is described as the output of an asymptotically stable time-invariant linear system fed by the input signal u whose transfer function $\Delta^\circ(z)$ satisfies

Assumption 2 *$\Delta^\circ(z)$ has bounded H-infinity norm: $\|\Delta^\circ\|_\infty \leq \delta$.*

Under the standard assumption that $u_t = 0$, $t < 0$, from Assumption 2 it easily follows (cf. [11]) that

$$\sum_{\tau=0}^{t} v_\tau^2 \leq \delta \sum_{\tau=0}^{t} u_\tau^2, \qquad (4)$$

which is fundamental in the derivation of the results stated in the sequel.
As for the noise signal, we assume that

Assumption 3 *w is a martingale difference sequence with respect to a filtration $\{\mathcal{F}_t\}$, satisfying:*

1. $\sup_t E[|w_t|^2/\mathcal{F}_{t-1}] < \infty$, almost surely (a.s);
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} w_t^2 = \sigma^2 > 0$ a.s.

Hence, differently from what is typically done in the switching control literature ([2, 4, 6, 7, 12]), we do not suppose that w is bounded.
We call model (3) with parameter ϑ° *nominal system*. Note that there might be in principle more than a nominal system, i.e., more than a value for ϑ° such that the assumption on the unmodeled dynamics term v is satisfied. What really matters for the further developments is that there exists *at least* one, to which we shall generically refer as ϑ°.

3 Supervisory control structure

We consider an *estimator-based* supervisor ([2, 4]) implemented as a structured hybrid dynamical system whose output σ is a switching signal taking values in the controller parameter set Γ and whose inputs are u and y. The supervisor is composed of two blocks: i) a *performance index generator*; and ii) a *switching logic* (see Figure 1). The hybrid nature of the system derives from the fact the switching logic part is an event-driven system, with the controller switching happening at the switching times. The interested reader is referred to [3] for an overview on different switching logics. We next describe our implementation of blocks i) and ii).

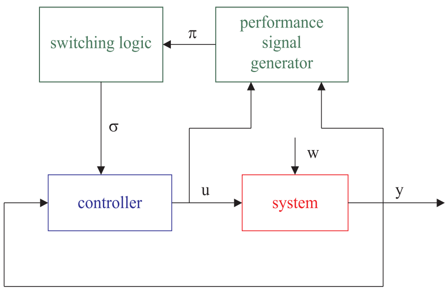

Figure 1: Supervisory control architecture.

3.1 Performance index generator

The performance index generator produces at each time instant $t \geq 0$ a signal $\pi_t(\vartheta)$, $\vartheta \in \Theta$, based on the input-output data collected from the system in the time interval $[0, t]$. π_t is then used to determine which model in the model set better resembles the system behavior. Here we use as performance index for model $\vartheta \in \Theta$ the standard least squares (LS) cost, which, on the basis of equation (3), can be expressed as ([13]): $\pi_t(\vartheta) = \sum_{s=0}^{t}(y_s - \varphi_{s-1}^T\vartheta)^2$. The best model at time t according to the LS cost is then the one with parameter

$$\hat{\vartheta}_t := \arg\min_{\vartheta\in\Theta} \pi_t(\vartheta).$$

3.2 Switching logic

The switching logic is an event-driven system which at the event times $\{t_i\}_{i=0,1,\dots}$, with $0 \leq t_0 < t_1 < \dots$, performs falsification of the currently operating controller and inference of the behavior of the candidate controllers when placed in feedback with the system ([14]). Here, according to the estimator-based approach, at each event time t_i both falsification and inference are performed based on the performance index π_{t_i}: *i)* the current controller is falsified if it is designed for a model with a parameter value ϑ that does not minimize π_{t_i}; *ii)* the controller to be switched in the loop is the one which is tuned to the model with parameter $\hat{\vartheta}_{t_i}$.

The switching signal σ_t takes values in the set Γ and represents the parameter of the controller placed in feedback with the system. If we define the parameter estimate ϑ_t as follows:

$$\vartheta_t = \begin{cases} \hat{\vartheta}_{t_i}, & \text{if } t = t_i,\ i = 0, 1, \dots \\ \vartheta_{t-1}, & \textit{otherwise}, \end{cases} \tag{5}$$

initialized with $\vartheta_{-1} = \bar{\vartheta} \in \Theta$, then, the switching signal is given by

$$\sigma_t = \Sigma(\vartheta_t). \tag{6}$$

We next explain how the event times are determined.

The idea underlying the estimator-based approach is that, as the amount of data collected from the system increases, the *estimated system*, i.e., the system governed by (3) with parameter ϑ_t, better resembles the system behavior. Hence, by imposing a desired behavior to the estimated system, one actually imposes that behavior to the underlying system (*self-tuning property*). On the other hand, if the control law were continuously tuned to the parameter estimate $\hat{\vartheta}_t$, i.e., $\vartheta_t = \hat{\vartheta}_t$, for all t, then the "frozen" estimated system dynamics will be stabilized, but the overall time-varying estimated system stability will not be ensured. A possible solution to this issue is then to update the parameter estimate at a slower rate than the updating of the system variables, so as to limit the estimated system time variability. In particular, one can make a *dwell time* elapse between two subsequent time instants when the estimate is updated. These instants are then the event times $\{t_i\}_{i=0,1,\dots}$. Those event times when the currently implemented controller is actually falsified are the *switching times*.

The *dwell time switching logic* has been successfully used in e.g. [2, 4, 15, 16]. Here, similarly to [16], we use a time-varying dwell time, adaptively selected based on the parameter estimate ϑ_t. Precisely, we first introduce the *dwell time function* $\tau_D : \Theta \to \mathbb{N}$ mapping each parameter $\vartheta \in \Theta$ in the dwell time $\tau_D(\vartheta) \in \mathbb{N}$, and, then, we define the event times recursively by

$$t_{i+1} = t_i + \tau_D(\vartheta_{t_i}), \qquad i = 0, 1, \dots$$

initialized with $t_0 = 0$.

We next design the dwell time function so as to stabilize the time-varying estimated system with parameter ϑ_t. Set $n := \max\{n_s, n_c\}$ and $m := \max\{m_s, m_c\}$ and let

$$x_t := [y_t \dots y_{t-(n-1)}\ u_{t-1} \dots u_{t-(m-1)}]^T. \tag{7}$$

Model (2) can then be given the representation

$$x_{t+1} = A(\vartheta)\, x_t + B(\vartheta) u_t + C w_{t+1}, \tag{8}$$

where

$$A(\vartheta) = \begin{bmatrix} a_1 & \dots & a_{n-1} & a_n & b_2 & \dots & b_{m-1} & b_m \\ 1 & 0 & \dots & & 0 & \dots & & 0 \\ & \ddots & \ddots & & & \ddots & & \\ & & 1 & 0 & & & & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & & 1 & 0 & \\ & \ddots & \ddots & & & & \ddots & \ddots \\ & & 0 & 0 & & & 1 & 0 \end{bmatrix},$$

$B(\vartheta) = [b_1\, 0\, \cdots\, 0\, 1\, 0\, \cdots\, 0]^T$, and $C = [1\, 0\, \cdots\, \cdots\, 0]^T$, with $a_i = 0$ if $i > n_s$, $b_i = 0$ if $i > m_s$. Note that this state space representation of model (2) is non minimal and, as it is easily seen from the block triangular matrix structure of $A(\vartheta)$, the added eigenvalues are all identically equal to zero.
By using vector x_t, controller (1) can be rewritten as

$$u_t = L(\gamma)x_t, \tag{9}$$

where $L(\gamma) = [s_0\, \cdots\, s_{n-1}\, s_n\, r_1\, \cdots\, r_{m-1}\, r_m]$, with $s_i = 0$ if $i > n_c$, and $r_i = 0$ if $i > m_c$.
The closed-loop system composed of model (8) controlled by (9) is then described by $x_{t+1} = F(\vartheta, \gamma)\, x_t + Cw_{t+1}$, where

$$F(\vartheta, \gamma) = A(\vartheta) + B(\vartheta)L(\gamma). \tag{10}$$

Fix a contraction constant μ, with $0 < \mu < 1$. Then, the dwell function is given by

$$\tau_D(\vartheta) := \inf\{\tau \in \mathbb{N} : \|F(\vartheta, \Sigma(\vartheta))^\tau\| \leq \mu\},$$

where $F(\vartheta, \Sigma(\vartheta))$ is obtained by replacing γ in (10) with $\Sigma(\vartheta)$. Note that the dwell function is well-defined since from Assumption 1 and the fact that the eigenvalues added in representation (8) are all equal to zero, it follows that $F(\vartheta, \Sigma(\vartheta))$ is stable for every $\vartheta \in \Theta$. Moreover, $F(\vartheta, \Sigma(\vartheta))$ with $\vartheta \in \Theta$ has all eigenvalues with absolute value smaller than the stability margin λ.
In the proposition below we show that this choice of the dwell function proves effective in stabilizing the estimated autonomous closed-loop system $x_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))\, x_t$. Before stating the proposition, we need to introduce some notations. Denote by $\mathcal{K}(P)$ the condition number with respect to the 2-norm of the square matrix P. From the stability margin condition it follows that $\bar{\mathcal{K}} := \sup_{\vartheta \in \Theta} \mathcal{K}(P_\vartheta)$, where P_ϑ is the solution to the Lyapunov equation associated with $\frac{1}{\lambda} F(\vartheta, \Sigma(\vartheta))$:

$$\frac{1}{\lambda}\, F(\vartheta, \Sigma(\vartheta))^T\, P\, \frac{1}{\lambda}\, F(\vartheta, \Sigma(\vartheta)) - P = -I,$$

is a bounded constant (cf. [11]).

Proposition 1

i) The adaptively selected dwell time interval is uniformly bounded in time: $\sup_{t \geq 0} \tau_D(\vartheta_t) \leq \bar{\tau}_D$, *with* $\bar{\tau}_D := \inf\{\tau \in \mathbb{N} : \sqrt{\bar{\mathcal{K}}}\, \lambda^\tau \leq \mu\}$.

ii) The autonomous estimated system $x_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))\, x_t$ *is exponentially stable, uniformly in time:* $\|x_t\| \leq \bar{\mathcal{K}} \nu^{t-t^\star} \|x_{t^\star}\|$, $0 \leq t^\star \leq t$, *with* $\nu = \max\{\lambda, \mu^{\frac{1}{\bar{\tau}_D}}\}$.

The proof of Proposition 1 is similar to the one of Proposition 3.1 in [16] and is omitted due to space limitations (see [11]).

4 Stability analysis

In this section, we prove that, if the unmodeled dynamics is sufficiently small, then stability is guaranteed by the proposed switching control scheme. In the case when the system belongs to the model class, we also show that the switching control scheme self-tunes.

Let us consider the *closed-loop estimated system*

$$\begin{cases} \hat{y}_{t+1} = [1 - \mathcal{A}(\vartheta_t; q^{-1})]\, \hat{y}_{t+1} + \mathcal{B}(\vartheta_t; q^{-1})\, \hat{u}_t + w_{t+1} \\ \hat{u}_t = \mathcal{S}(\Sigma(\vartheta_t); q^{-1})\, \hat{y}_t + [1 - \mathcal{R}(\Sigma(\vartheta_t); q^{-1})]\, \hat{u}_t. \end{cases}$$

It can be easily seen that the *switching control system*

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta^\circ; q^{-1})]\, y_{t+1} + \mathcal{B}(\vartheta^\circ; q^{-1})\, u_t + v_t + w_{t+1} \\ u_t = \mathcal{S}(\sigma_t; q^{-1})\, y_t + [1 - \mathcal{R}(\sigma_t; q^{-1})]\, u_t, \end{cases}$$

with σ_t given by (6), can be represented as a variational system with respect to the closed-loop estimated system as follows:

$$\begin{cases} y_{t+1} = [1 - \mathcal{A}(\vartheta_t; q^{-1})] y_{t+1} + \mathcal{B}(\vartheta_t; q^{-1})\, u_t + v_t + w_{t+1} + e_t \\ u_t = \mathcal{S}(\Sigma(\vartheta_t); q^{-1})\, y_t + [1 - \mathcal{R}(\Sigma(\vartheta_t); q^{-1})]\, u_t, \end{cases}$$

where $e_t := \varphi_t^T [\vartheta^\circ - \vartheta_t]$ is the *estimation error.*
The switching control system stability then can be proven based on the fact that, on the one hand, by adopting a slow switching, uniform exponential stability of the closed-loop estimated system is guaranteed (cf. Proposition 1); on the other hand, by switching to the controller designed for the best LS model, one keeps the internally generated perturbation term e_t 'small'. This last property is actually shown in Proposition 2 below. The proofs of Proposition 2 and the instrumental result in Theorem 1 are omitted due to space limitations and can be found in the technical report [11].

Set $d := \max_{\vartheta_1, \vartheta_2 \in \Theta} \|\vartheta_1 - \vartheta_2\|$. By using definition (5), ϑ_t can be shown to satisfy the following property.

Theorem 1 *Suppose that* u_t *is* $\mathcal{F}_t$*-measurable. Then,*

$$\begin{aligned} &(\vartheta^\circ - \vartheta_{t_i})^T \sum_{s=1}^{t_i} \varphi_{s-1} \varphi_{s-1}^T (\vartheta^\circ - \vartheta_{t_i}) \\ &\leq o\Big(\sum_{s=1}^{t_i} \|\varphi_{s-1}\|^2\Big) + 2d\sqrt{2\delta} \sum_{s=1}^{t_i} \|\varphi_{s-1}\|^2, \quad a.s. \end{aligned}$$

By a suitable manipulation of the sole result in Theorem 1, we get the following bound on the estimation error.

Proposition 2 *Suppose that* u_t *is* $\mathcal{F}_t$*-measurable. Then, for every* $0 < \epsilon < d$,

$$\sum_{t=0, t \notin \mathcal{B}_{N-1}}^{N-1} e_t^2 \leq c(\epsilon, \delta) \sum_{t=0}^{N-1} \|\varphi_t\|^2 + o\Big(\sum_{t=0}^{N-1} \|\varphi_t\|^2\Big), \tag{11}$$

a.s., where $\mathcal{B}_{N-1}$ is a set of instant points which depends on N, whose cardinality is uniformly bounded: $|\mathcal{B}_{N-1}| \leq C_{\mathcal{B}}, \forall N$, and $c(\epsilon,\delta) := c_1(\epsilon) + \frac{1}{c_2(\epsilon)}\sqrt{2\delta}$, with $c_1(\epsilon)$ and $c_2(\epsilon)$ smooth functions of ϵ, $0 < \epsilon < d$, which tend to zero as $\epsilon \to 0$.

Remark: If the unmodeled dynamics is not present, i.e., $\delta = 0$, then equation (11) translates into $\frac{1}{N}\sum_{t=0,\ t\notin\mathcal{B}_{N-1}}^{N-1} e_t^2 = o\Big(\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^2\Big)$, which entails that the average square estimation error vanishes when stability is ensured. This is a key property for the switching scheme to self-tune.

We are now in the position to prove the stability result.

Theorem 2

i) If $\delta \leq \delta^*$, where $\delta^* > 0$ is a constant depending on the parameter uncertainty set structure, the control map Σ, and the contraction constant μ, then the switching control system is stable:

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\left[y_t^2+u_t^2\right] < \infty, \qquad a.s.$$

ii) If $\delta = 0$, then self-tuning is achieved:

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\left[y_t^2+u_t^2\right] = \limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\left[\hat{y}_t^2+\hat{u}_t^2\right],$$

a.s., where $\hat{y}$ and $\hat{u}$ are the output and input signals generated by the closed-loop estimated system.

Proof: i) Fix a time point $N > 0$.
In view of the variational representation of the switching control system it is easily seen that the evolution in time of x_t in (7) is governed by

$$\begin{aligned} x_{t+1} &= F(\vartheta^\circ, \Sigma(\vartheta_t))\, x_t + C[v_t + w_{t+1}] &(12)\\ &= F(\vartheta_t, \Sigma(\vartheta_t))\, x_t + C[v_t + w_{t+1} + e_t], &(13)\end{aligned}$$

where $F(\vartheta,\gamma)$ is defined by (10) and $C = [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0]^T$. Consider set $\mathcal{B}_{N-1}$ introduced in Proposition 2. For the following derivations, it is convenient to use (12) in the time instants $t \in \mathcal{B}_{N-1}$ and (13) for $t \notin \mathcal{B}_{N-1}$, thus getting

$$x_{t+1} = \begin{cases} F(\vartheta^\circ, \Sigma(\vartheta_t))\, x_t + C[v_t + w_{t+1}], t \in \mathcal{B}_{N-1} \\ F(\vartheta_t, \Sigma(\vartheta_t))\, x_t + C[v_t + e_t + w_{t+1}], t \notin \mathcal{B}_{N-1}. \end{cases}$$

Note now that $\|F(\vartheta^\circ, \Sigma(\vartheta_t))\|$ is uniformly bounded. By the uniform exponential stability of $x_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))x_t$ (Proposition 1), and the uniform boundedness of $|\mathcal{B}_{N-1}|$ (see Proposition 2), it is then easily shown that x_t can be bounded as follows

$$\|x_t\| \leq k_1\Big\{\nu^t\|x_0\| + \sum_{i=0,i\notin\mathcal{B}_{N-1}}^{t-1}\nu^{t-i}|v_i + w_{i+1} + e_i|\Big\},$$

$t \leq N$, where k_1 is a suitable constant, independent of N, and ν is the constant introduced in Proposition 1. Some cumbersome computations then lead to

$$\|x_t\|^2 \leq 4k_1^2\Big\{\nu^{2t}\|x_0\|^2 + \frac{1}{1-\nu}\Big[\sum_{i=1}^{t}\nu^{t-i}w_i^2 + \sum_{i=0}^{t-1}\nu^{t-i}v_i^2 + \sum_{i=0,i\notin\mathcal{B}_{N-1}}^{t-1}\nu^{t-i}e_i^2\Big]\Big\}, \qquad (14)$$

$t \leq N$, from which we finally get

$$\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2 \leq k_2\Big[\frac{1}{N}\sum_{t=1}^{N}w_i^2 + \frac{1}{N}\sum_{t=0}^{N-1}v_i^2 + \frac{1}{N}\sum_{t=0,t\notin\mathcal{B}_{N-1}}^{N-1}e_t^2\Big] + o(1), \qquad (15)$$

k_2 being a suitable constant.
We next bound the terms in the right-hand side of (15). By Assumption 3, $\frac{1}{N}\sum_{t=1}^{N}w_t^2 = O(1)$. By equation (4) and the definition of φ_t, $\frac{1}{N}\sum_{t=0}^{N-1}v_t^2 \leq \delta\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^2$. Term $\frac{1}{N}\sum_{t=0,t\notin\mathcal{B}_{N-1}}^{N-1}e_t^2$ is immediately bounded by means of (11). Hence, the final bound for $\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2$ is obtained: $\frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2 \leq h(\epsilon,\delta)\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^2 + o\Big(\sum_{t=0}^{N-1}\|\varphi_t\|^2\Big) + O(1)$, $a.s.$, where we set $h(\epsilon,\delta) := k_2[c(\epsilon,\delta)+\delta]$. Since $\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^2 \leq \frac{1}{N}\sum_{t=0}^{N}\|x_t\|^2$, we then have that for all δ for which there exists ϵ with $0 < \epsilon < d$ such that $h(\epsilon,\delta) < 1$, $\frac{1}{N}\sum_{t=0}^{N}\|x_t\|$ (and hence $\frac{1}{N}\sum_{t=0}^{N-1}\|\varphi_t\|^2$) remains bounded, i.e., the system is stable. By recalling the definition of $c(\epsilon,\delta)$ in Proposition 2, we have that if $\delta = 0$, then $h(\epsilon, 0) = c_1(\epsilon)$, and hence stability is ensured by taking ϵ sufficiently small so that $c_1(\epsilon) < 1$. In the presence of unmodeled dynamics, stability is ensured if δ is smaller or equal to δ^* determined by computing the infimum of $h(\epsilon,\delta)$ with respect to ϵ, say $\bar{h}(\delta)$, and then imposing $\bar{h}(\delta^*) < 1$. For continuity reasons, such a δ^* exists and satisfies $\delta^* > 0$.

ii) Suppose that there is no unmodeled dynamics ($v_t = 0$). We next prove that y_t and $\hat{y}_t$ satisfy

$$y_t^2 = o(t) \text{ and } \hat{y}_t^2 = o(t). \qquad (16)$$

As shown in (14) with N set equal to t, for any $t > 0$,

$$\|x_t\|^2 \leq k_3\left\{\nu^{2t}\|x_0\|^2 + \sum_{i=1}^{t}\nu^{t-i}w_i^2 + \sum_{i=0,i\notin\mathcal{B}_{t-1}}^{t-1}\nu^{t-i}e_i^2\right\},$$

where $|\mathcal{B}_{t-1}| \leq C_{\mathcal{B}}, \forall t$. As for the first term, $\nu^{2t}\|x_0\| = o(t)$. As for the second term, from Assumption 3 it follows that $\sum_{i=1}^{t}\nu^{t-i}w_i^2 = o(t)$ (cf. [11]). As for the third term, by using Proposition 2 with $\delta = 0$ and the L^2-stability result in point i) we get

$$\sum_{i=0,i\notin\mathcal{B}_{t-1}}^{t-1} e_i^2 = o\Big(\sum_{i=0}^{t-1}\|\varphi_i\|^2\Big) = o(t). \qquad (17)$$

Therefore, by using these estimates we get $\|x_t\|^2 = o(t)$ from which (16) immediately follows for y_t.

As for $\hat{y}_t$, the closed-loop estimated system can be put in the state space form $\hat{x}_{t+1} = F(\vartheta_t, \Sigma(\vartheta_t))\,\hat{x}_t + Cw_{t+1}$, where $\hat{x}_t := [\hat{y}_t \ldots \hat{y}_{t-(n-1)}\,\hat{u}_{t-1} \ldots \hat{u}_{t-(m-1)}]^T$. Then (16) for $\hat{y}$ can be derived by an analogous —even though simpler— proof to the one for the output y of the control switching system, since in this case the perturbation term e_t is not present.

Set $p_t := [y_t\ u_{t-1}]^T$ and $\hat{p}_t := [\hat{y}_t\ \hat{u}_{t-1}]$. Then, we prove point ii) by showing that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (\|p_t\|^2 - \|\hat{p}_t\|^2) = 0. \qquad (18)$$

Indeed, $\lim_{N\to\infty}\{\frac{1}{N}\sum_{t=0}^{N-1}[\,y_t^2 + u_t^2\,] - \frac{1}{N}\sum_{t=0}^{N-1}[\,\hat{y}_t^2 + \hat{u}_t^2\,]\} = \lim_{N\to\infty}\{\frac{1}{N}\sum_{t=1}^{N}(\|p_t\|^2 - \|\hat{p}_t\|^2) - \frac{1}{N}(y_N^2 - \hat{y}_N^2) + \frac{1}{N}(y_0^2 - \hat{y}_0^2)\}$, where $\lim_{N\to\infty}\frac{1}{N}(y_0^2 - \hat{y}_0^2) = 0$ and $\lim_{N\to\infty}\frac{1}{N}(y_N^2 - \hat{y}_N^2) = 0$ by equation (16).

Consider now the following inequality

$$\left|\frac{1}{N}\sum_{t=1}^{N}\left(\|p_t\|^2 - \|\hat{p}_t\|^2\right)\right| \le \left[\frac{1}{N}\sum_{t=1}^{N}\left(\|p_t\| + \|\hat{p}_t\|\right)^2\right]^{\frac{1}{2}} \left[\frac{1}{N}\sum_{t=1}^{N}\left(\|p_t\| - \|\hat{p}_t\|\right)^2\right]^{\frac{1}{2}}. \qquad (19)$$

The first factor is bounded. Indeed,

$$\left[\frac{1}{N}\sum_{t=1}^{N}(\|p_t\| + \|\hat{p}_t\|)^2\right]^{\frac{1}{2}} \le \frac{\sqrt{2}}{N}\left[\sum_{t=1}^{N}\|p_t\|^2 + \sum_{t=1}^{N}\|\hat{p}_t\|^2\right]^{\frac{1}{2}},$$

where $\frac{1}{N}\sum_{t=1}^{N}\|p_t\|^2$ is bounded as a consequence of the L^2-stability result i), whereas the boundedness of $\frac{1}{N}\sum_{t=1}^{N}\|\hat{p}_t\|^2$ can be proven similarly.

As for the second factor, we have

$$\frac{1}{N}\sum_{t=1}^{N}(\|p_t\| - \|\hat{p}_t\|)^2 \le \frac{1}{N}\sum_{t=1}^{N}\|p_t - \hat{p}_t\|^2. \qquad (20)$$

Observe that $x_t - \hat{x}_t$ is governed by the following equation $(x_{t+1} - \hat{x}_{t+1}) = F(\vartheta_t, \Sigma(\vartheta_t))\,(x_t - \hat{x}_t) + Ce_t$. Then by an argument similar to the one leading to equation (15), we get that $\frac{1}{N}\sum_{t=0}^{N}\|x_t - \hat{x}_t\|^2 \le k_2\frac{1}{N}\sum_{t=0, t\notin\mathcal{B}_{N-1}}^{N-1} e_t^2 + o(1)$. Since $\|p_t - \hat{p}_t\| \le \|x_t - \hat{x}_t\|$, from this equation and equation (17), we have $\limsup_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N}\|p_t - \hat{p}_t\|^2 = 0$. Then, taking into account the boundedness of $\frac{1}{N}\sum_{t=1}^{N}(\|p_t\| + \|\hat{p}_t\|)^2$ and inequality (20), from equation (19) the conclusion is finally drawn that equation (18) holds true. ■

5 Concluding remarks

In this paper we have described a switching control scheme for a discrete time linear system affected by a possibly unbounded stochastic noise. We have proved that such a scheme is effective in stabilizing the system also when it is designed based on a reduced order model of the system. This requires that the unmodeled dynamics is sufficiently small. If there is no unmodeled dynamics, then the switching scheme also self-tunes.

The bound on the admissible unmodeled dynamics depends on the parameter uncertainty set structure and on some design parameters (the stability margin λ and the contraction constant μ). Further investigation is needed to evaluate the conservativeness of the bound.

References

[1] R. Middleton, G. Goodwin, D. Hill, and D. Mayne, "Design issues in adaptive control," *IEEE Trans. on Automatic Control*, vol. AC-33, pp. 50–58, 1988.

[2] A. Morse, "Supervisory control of families of linear set-point controllesr—Part 1: Exact matching," *IEEE Trans. on Automatic Control*, vol. AC-41, pp. 1413–1431, 1996.

[3] A. Morse, "Control using logic-based switching," in *Trends in Control* (A. Isidori, ed.), pp. 69–113, New York: Springer-Verlag, 1995.

[4] A. Morse, "Supervisory control of families of linear set-point controllesr—Part 2: Robustness," *IEEE Trans. on Automatic Control*, vol. AC-42, pp. 1500–1515, 1997.

[5] J. Hespanha, *Logic-based switching algorithms in control.* PhD thesis, Dept. of Electrical Engineering, Yale University, 1998.

[6] F. Pait and F. Kassab Jr, "On a class of switched, robustly stable, adaptive systems," in *Proc. 38th Conf. on Decision and Control*, pp. 2673–2678, December 1999.

[7] D. Borrelli, A. Morse, and E. Mosca, "Discrete-time supervisory control of families of 2-DOF linear set-point controllers," *IEEE Trans. on Automatic Control*, vol. AC-44, pp. 178–181, 1999.

[8] B. O. Anderson, T. Brinsmead, F. de Bruyne, J. Hespanha, D. Liberzon, and A. S. Morse, "Multiple model adaptive control. I: Finite controller coverings," *Int. J. of Robust and Nonlinear Control*, vol. 10, pp. 909–929, 2000. George Zames Special Issue.

[9] E. Mosca and T. Agnoloni, "Controller selection in switching supervisory control under a coarse candidate model distribution," in *Proc. 38th Conf. on Decision and Control*, pp. 1333–1338, December 1999.

[10] D. Borrelli, E. Mosca, and J. Zhang, "Sidestepping the singularity problem via self-excitation in adaptive 2-DOF control," in *Proc. 3rd Europ. Control Conf.*, pp. 885–890, September 1995.

[11] M. Prandini and M. Campi, "Logic-based switching for the stabilization of stochastic systems in presence of unmodeled dynamics," Tech. Rep. 2001-01-23, University of Brescia, Brescia, Italy, March 2001.

[12] J. Hespanha, D. Liberzon, A. S. Morse, B. O. Anderson, T. Brinsmead, and F. de Bruyne, "Multiple model adaptive control. II: Switching," *Int. J. of Robust and Nonlinear Control*, vol. 11, no. 5, pp. 479–496, 2001.

[13] L. Ljung, *System Identification: Theory for the User.* Prentice Hall, Englewood Cliffs, NJ, 1999.

[14] M. Safonov and T. Tsao, "The unfalsified control concept and learning," *IEEE Trans. on Automatic Control*, vol. AC-42, pp. 843–847, 1997.

[15] M. Prandini, S. Bittanti, and M. Campi, "A penalized identification criterion for securing controllability in adaptive control," *J. of Math. Syst., Estim. and Control*, vol. 8, pp. 491–494, 1998. Full paper electronic access code: 29460.

[16] M. Prandini and M. Campi, "Adaptive LQG control of input-output systems - A cost-biased approach," *SIAM J. Control and Optim.*, vol. 39, no. 5, pp. 1499–1519, 2001.