RANDOMIZATION
in SYSTEMS and CONTROL:
a CHANGE of PERSPECTIVE

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with …

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Algo Care’

outline:

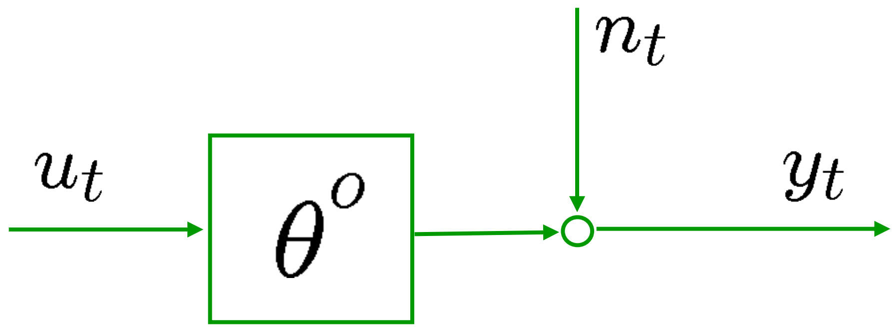
(i) focusing on randomization:
an example in estimation theory

(ii) what is a randomized algorithm?

(iii) where are we?

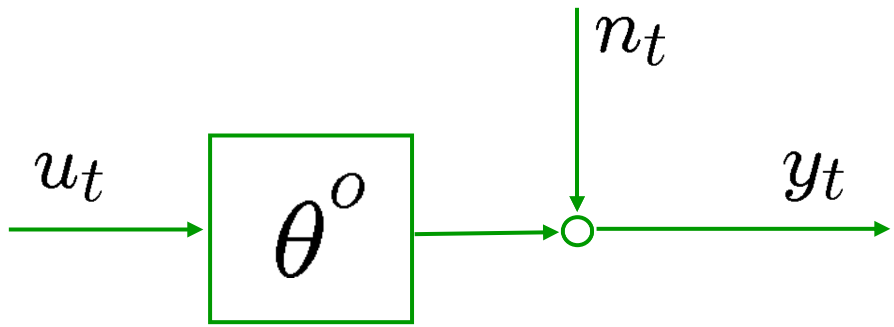
(iv) uncertain systems

an estimation problem



$$y_t = \theta^o u_t + n_t$$

an estimation problem



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- select $u_t, \ t = 1, \ldots, N$
- measure y_t

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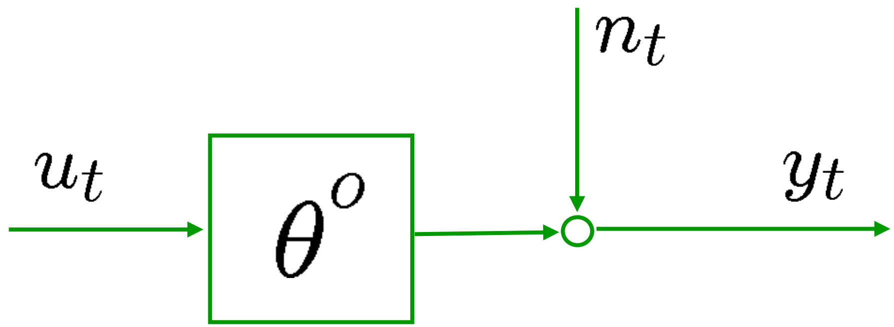

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goal: provide an interval Θ for θ^o

$$M\colon \ (u_t, y_t) \ \rightarrow \ \Theta$$

an estimation problem

<u>requirements:</u>

an estimation problem

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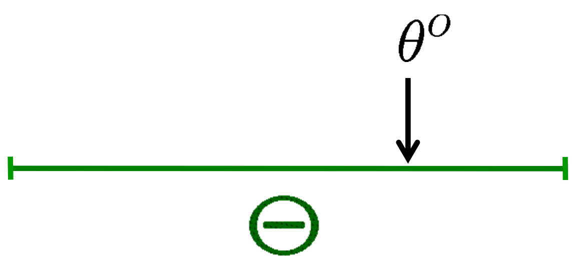
(i) no assumptions on n_t

an estimation problem

requirements:

(i) no assumptions on n_t

(ii) we want to issue a certificate of reliability on Θ valid $\forall n_t$



is this at all possible?

. . . no deterministic algorithm solves the problem

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- take an input sequence u_t
- measure y_t

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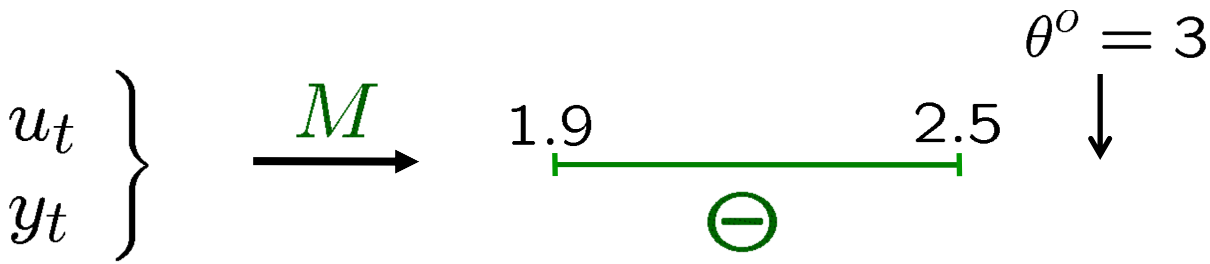

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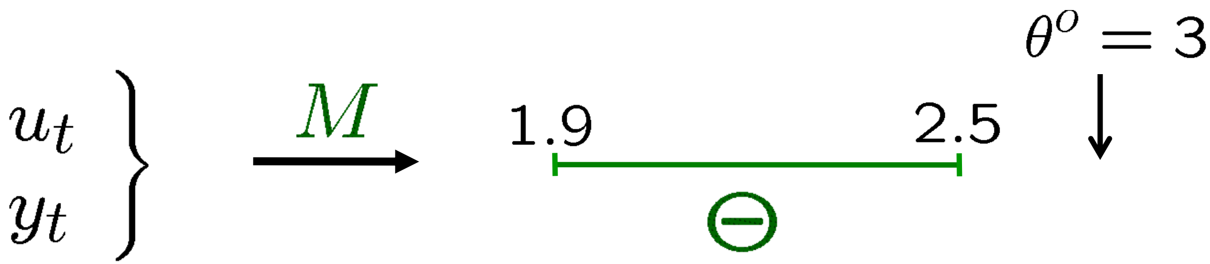

$$y_t = 3u_t + n_t$$

$$\Downarrow$$

$$n_t = -3u_t + y_t$$

- a deterministic algorithm that comes with a certificate of reliability does not exist!

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- shall we give up?

moving one step ahead …
a change of perspective

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a change of perspective

Algorithm

1) $$u_t = \begin{cases} +1, & \text{with prob} = 0.5 \\ -1, & \text{with prob} = 0.5 \end{cases}$$

(R. Fisher)

moving one step ahead …
a change of perspective

Algorithm

1) $u_t = \begin{cases} +1, & \text{with prob} = 0.5 \\ -1, & \text{with prob} = 0.5 \end{cases}$ (R. Fisher)

2) M: $(u_t, y_t) \longrightarrow \ominus$ needs some explanation

(work done with E. Weyer)

explaining M

number of observations $= 7$

explaining M

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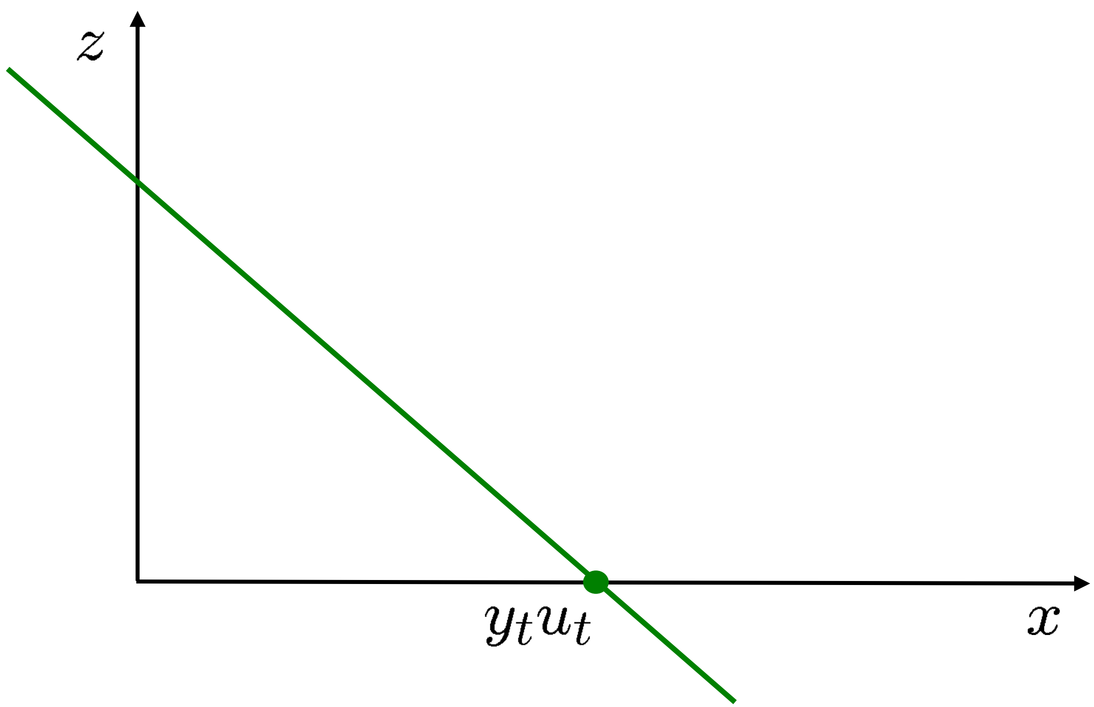
$z = y_t \cdot u_t - x$ -45 degrees line $t = 1, 2, \ldots, 7$



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construct some averages:

$$\frac{1}{|T_1|} \sum_{t \in T_1} [y_t \cdot u_t - x]$$

$$\frac{1}{|T_2|} \sum_{t \in T_2} [y_t \cdot u_t - x]$$

$$\vdots$$

explaining M

numerical example:

u_t	-1	1	1	1	-1	1	-1
y_t	-0.45	0.97	1.74	1.80	0.1	1.92	-0.73

explaining M

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u_t	-1	1	1	1	-1	1	-1
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	1	2	3	4	5	6	7
T_1	●	●	○	●	●	○	○
T_2	●	○	●	●	○	●	○
T_3	○	●	●	○	●	●	○
T_4	●	●	○	○	○	●	●
T_5	●	○	●	○	●	○	●
T_6	○	●	●	●	○	○	●
T_7	○	○	○	●	●	●	●

$$\Longrightarrow \frac{1}{4} \sum_{t \in \{1,2,4,5\}} [y_t \cdot u_t - x]$$

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etc.

explaining M

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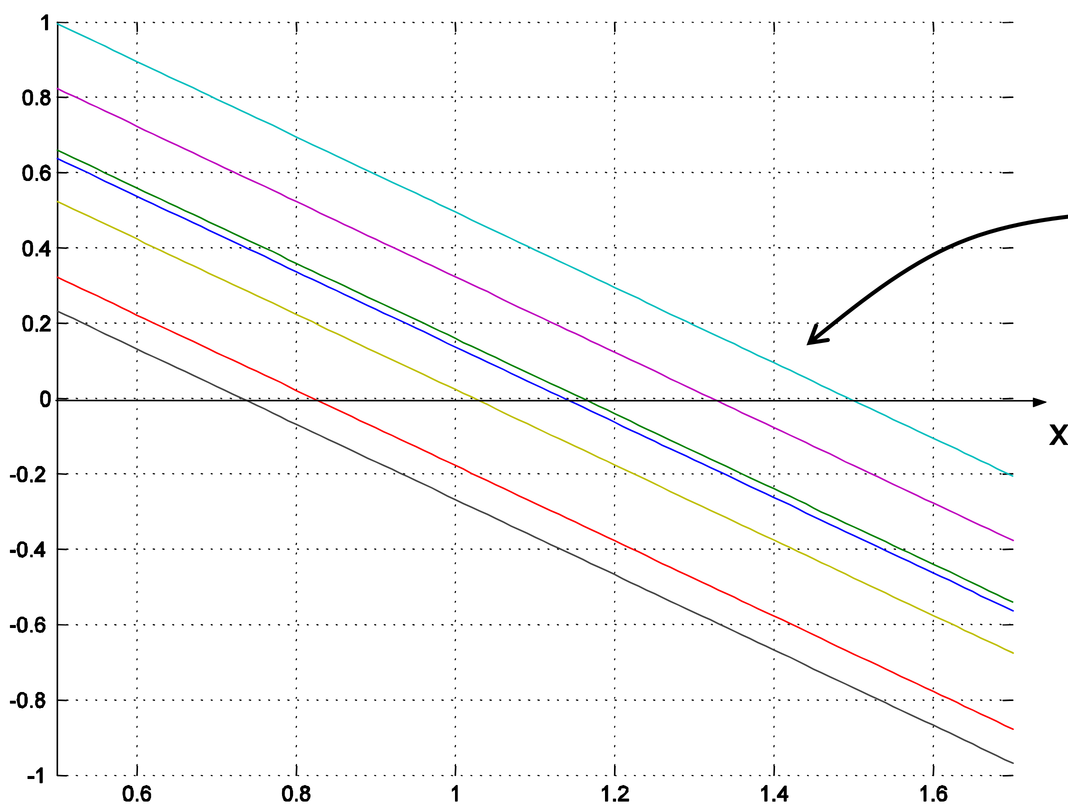
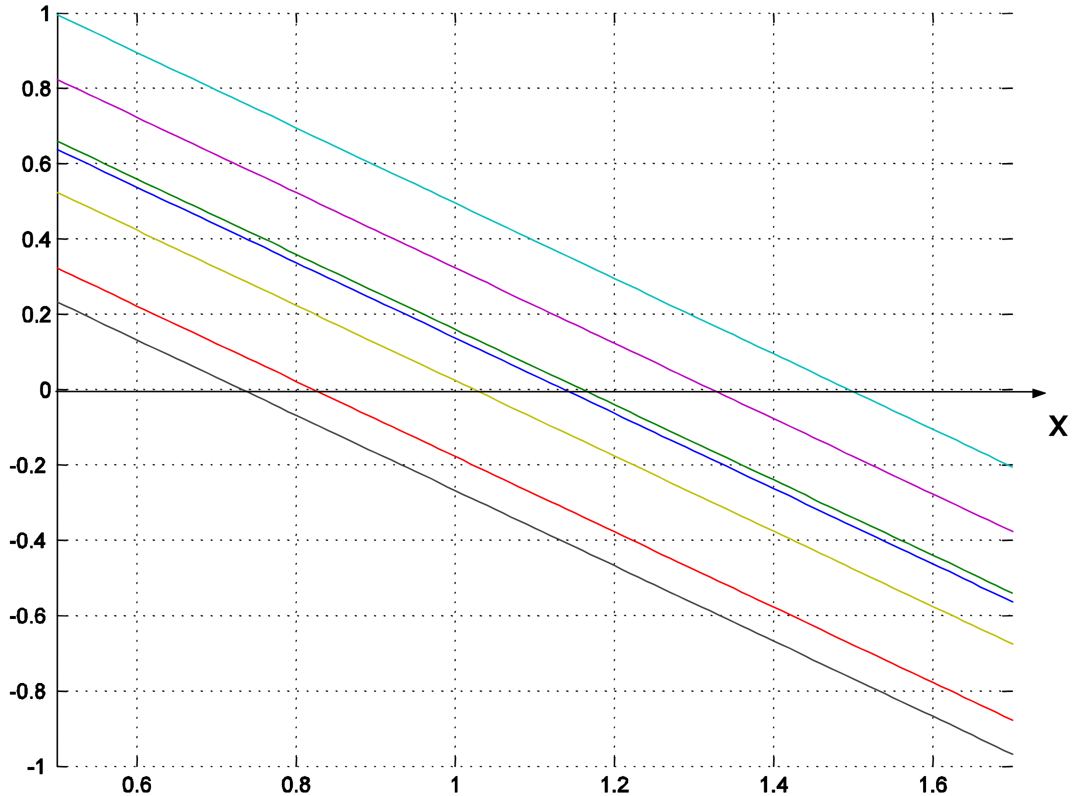


$\frac{1}{4}\sum_{t\in T}[y_t \cdot u_t - x]$

explaining M

$$\frac{1}{4}\sum_{t\in\{1,2,4,5\}}[y_t\cdot u_t - x] = [\theta^o - x] + \frac{1}{4}\sum_{t\in\{1,2,4,5\}} n_t\cdot u_t$$

$y_t = \theta^o u_t + n_t$



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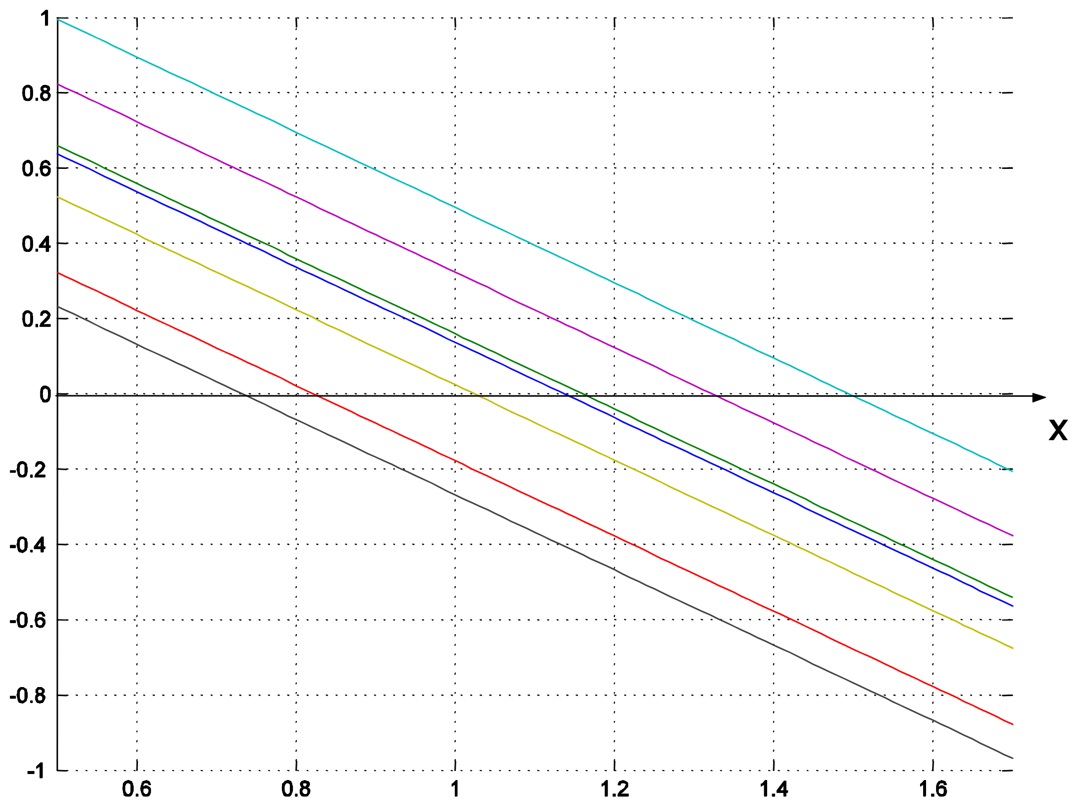

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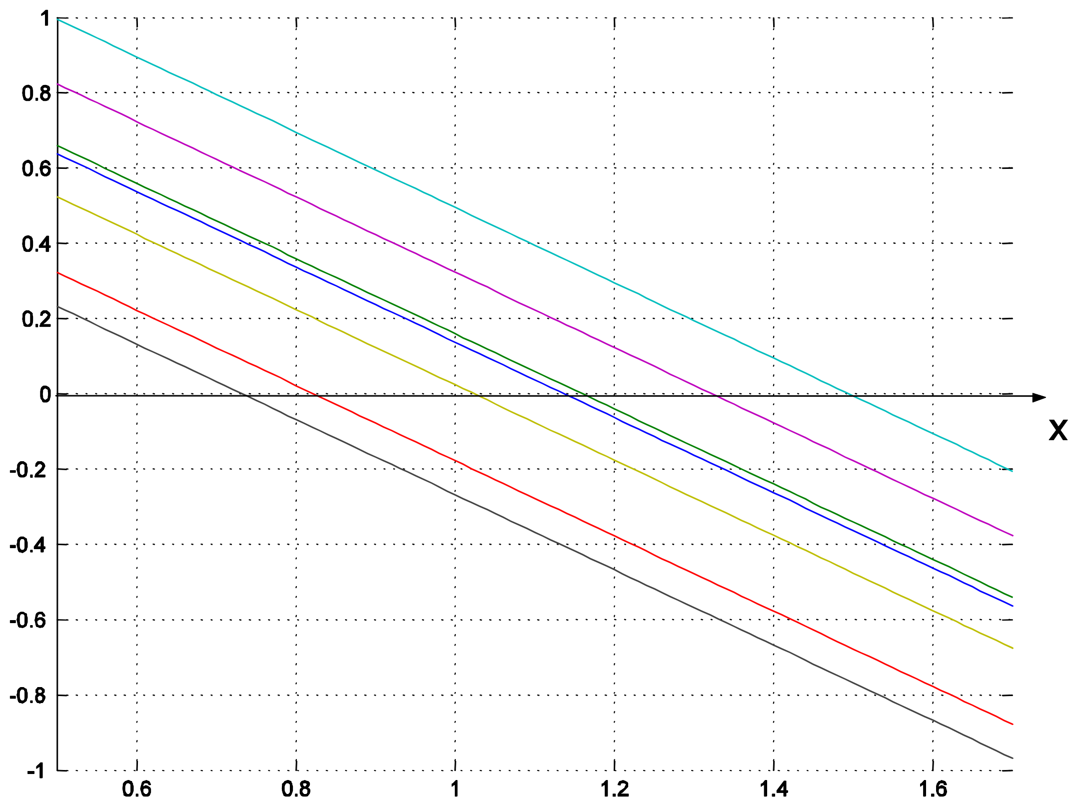
$\theta^o u_t + n_t$



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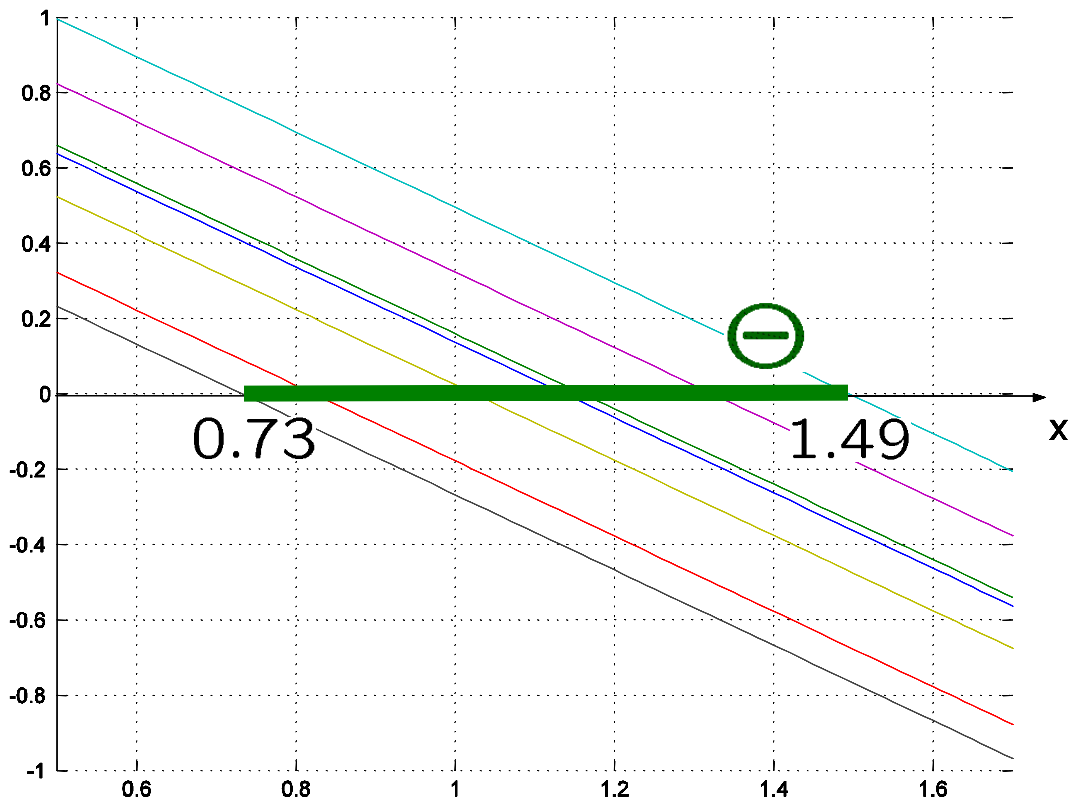
⊖ = not all functions are positive or all are negative



explaining M

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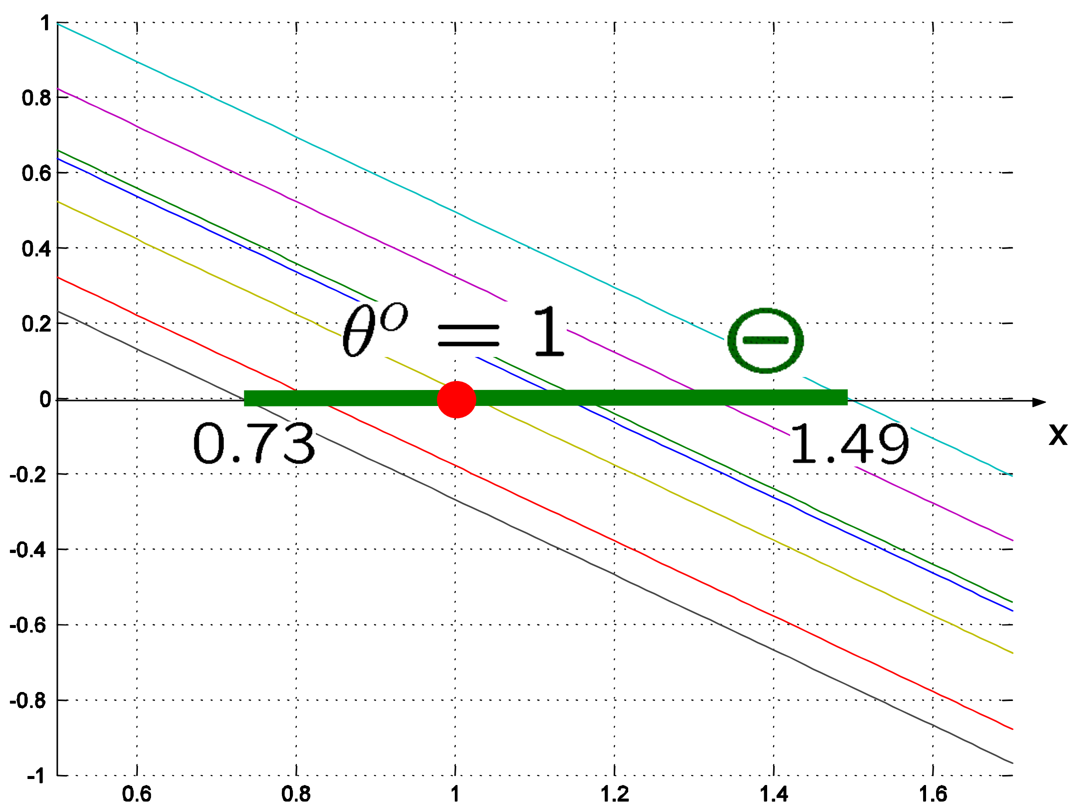

Theorem (with E. Weyer)

$\forall n_t, \quad \theta^o \in \Theta$ with probability 0.75.

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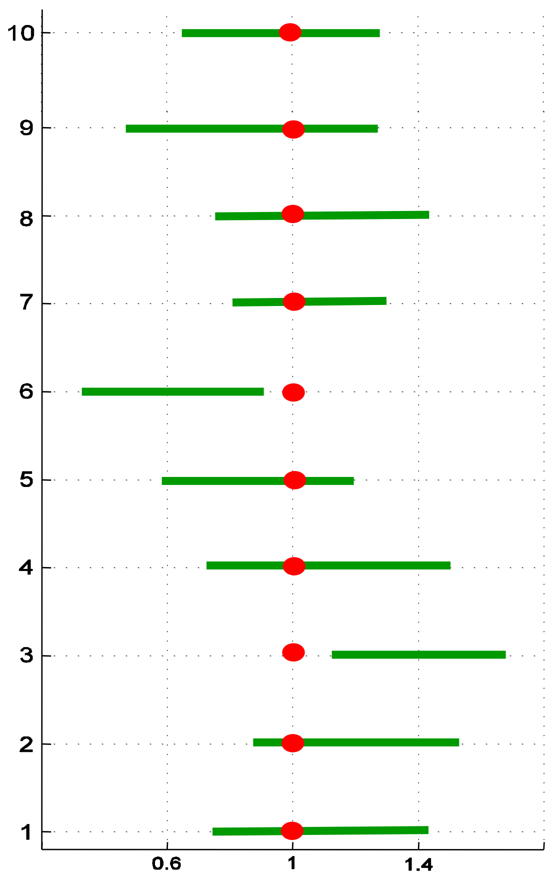

n_t is fixed

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$N = 7$

$N = 1023$

$\Theta = [0.945, 1.033]$ with probability $= 0.975$

Theorem (with E. Weyer)

$\forall n_t, \quad \theta^o \in \Theta$ with probability 0.75.

Theorem (consistency)

Under general assumptions,

$\Theta \longrightarrow \{\theta^o\}$ as $N \rightarrow \infty$

in summary:

for any n_t:

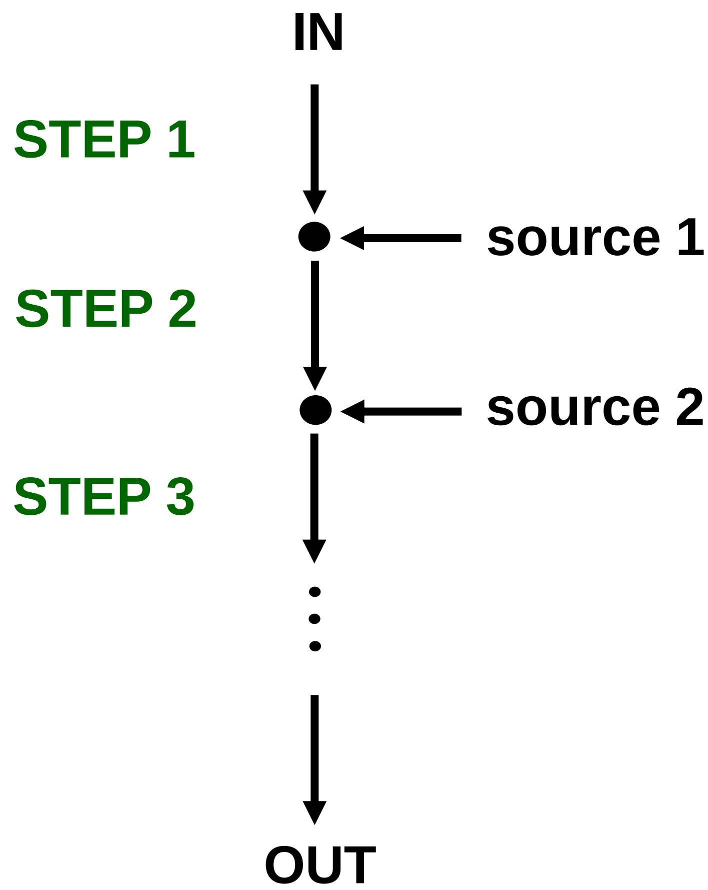
(i) $\theta^o \in \Theta$ with a precise probability

(ii) the size of Θ depends on the strength of the noise

(iii) Θ shrinks around θ^o as *N* increases how is all this possible?

what is a randomized algorithm?

deterministic algorithm



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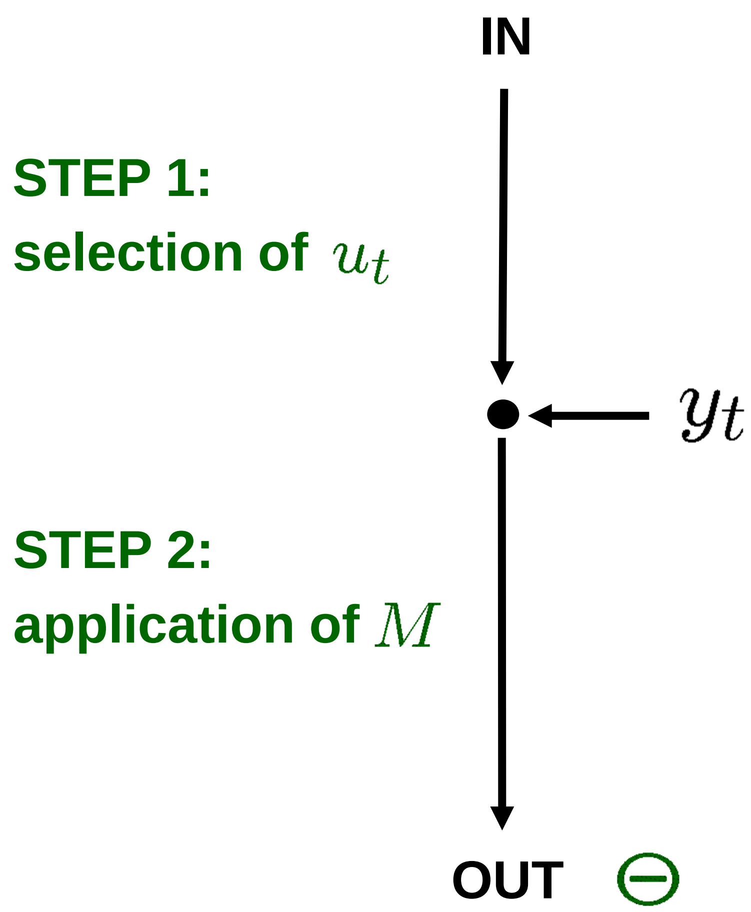
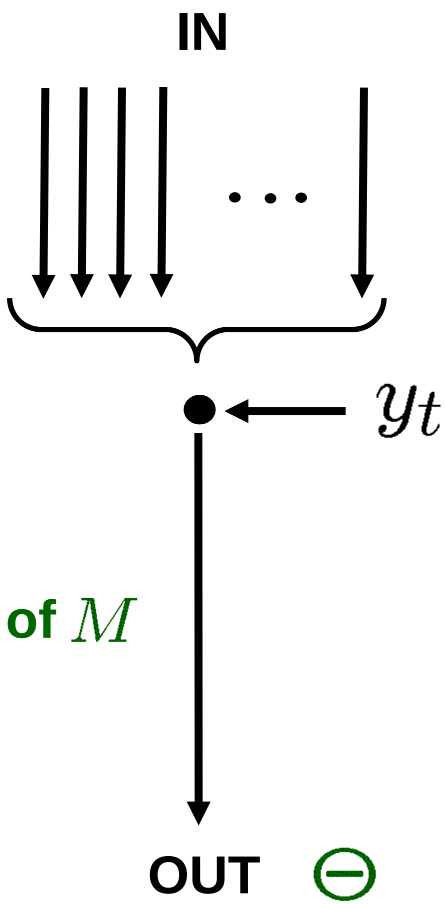


randomized algorithm

STEP 1:
random selection of u_t



STEP 2:
application of M

randomized algorithm

a randomized algorithm is an algorithm where one or more steps are based on a random choice

that is – among many deterministic choices – one choice is selected at random according to a probability *P*

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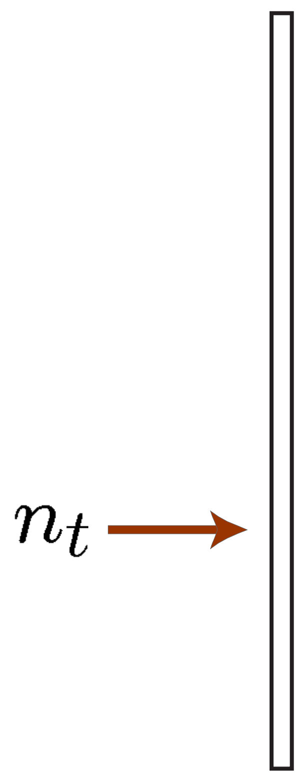
why is this useful?

deterministic vs. randomized

deterministic vs. randomized



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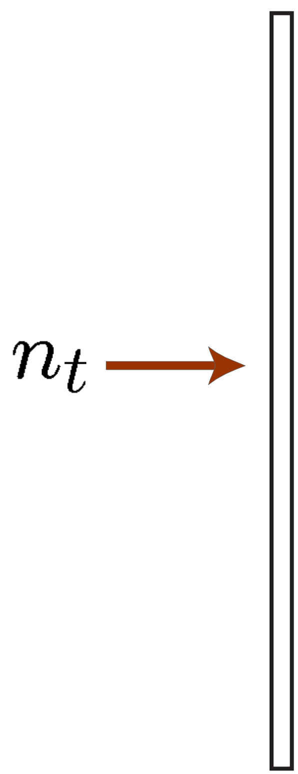

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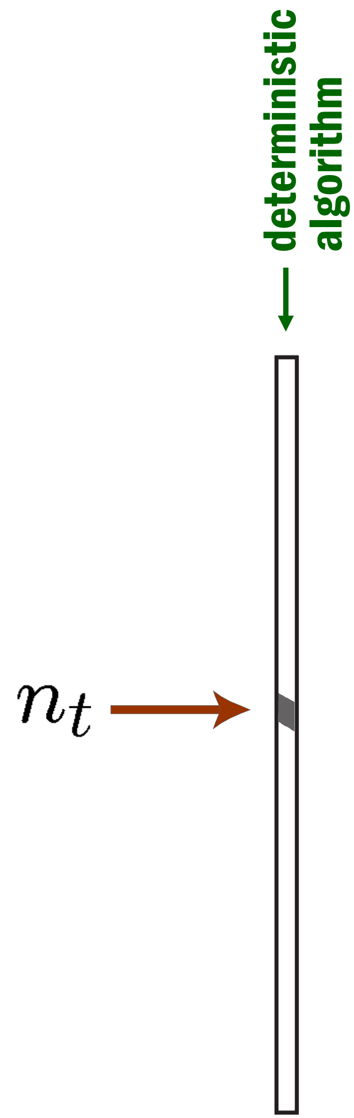

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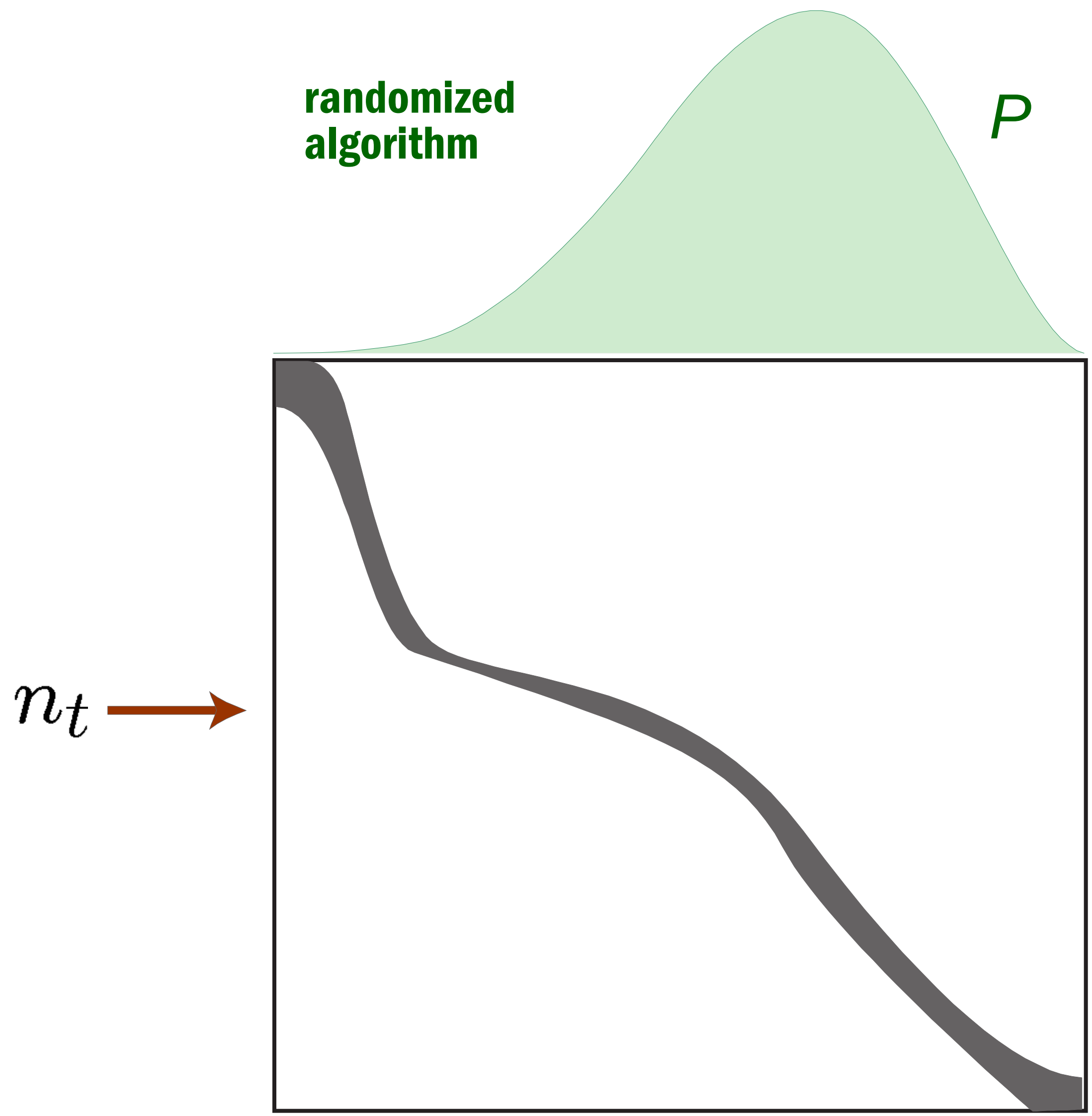

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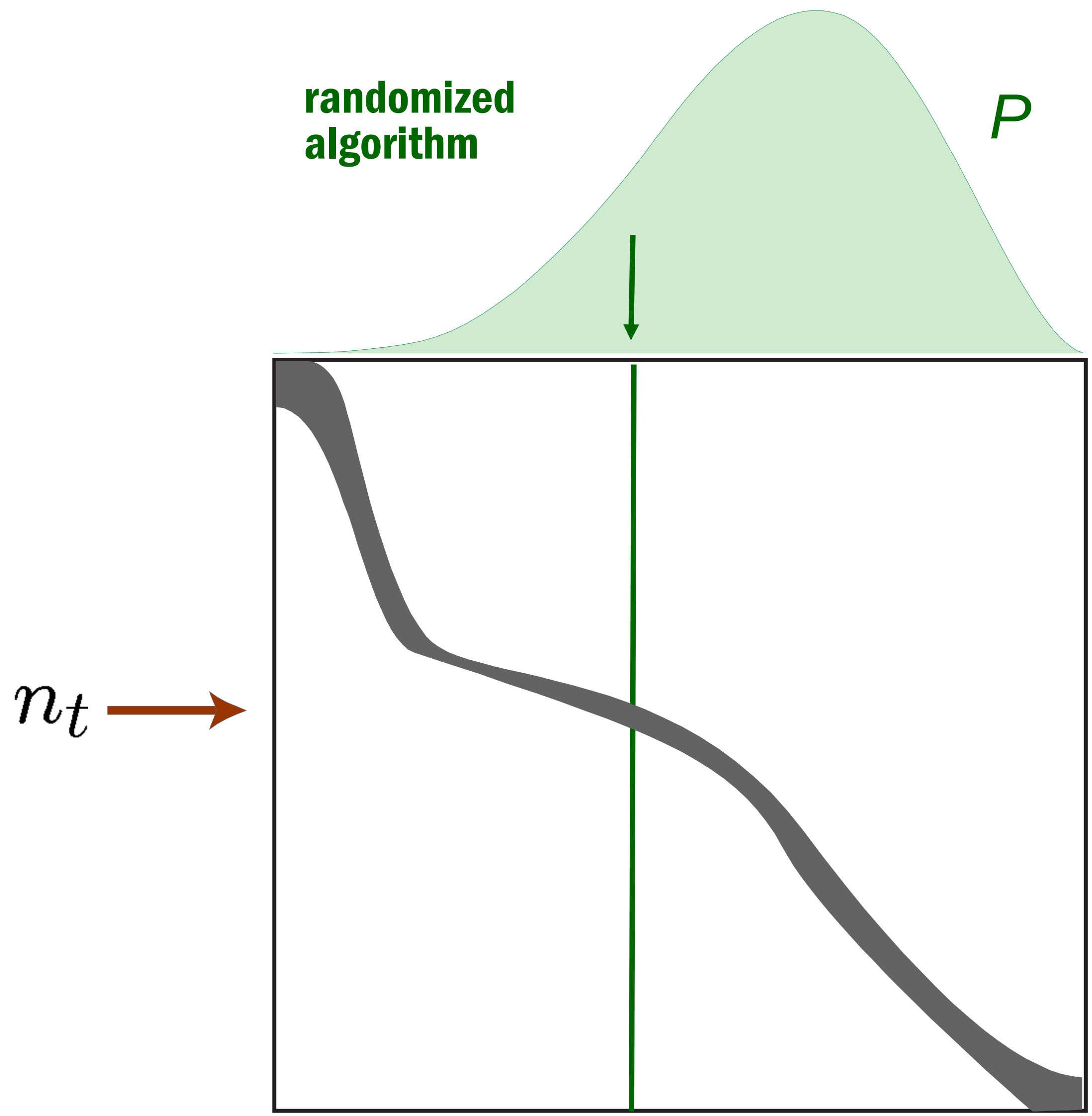

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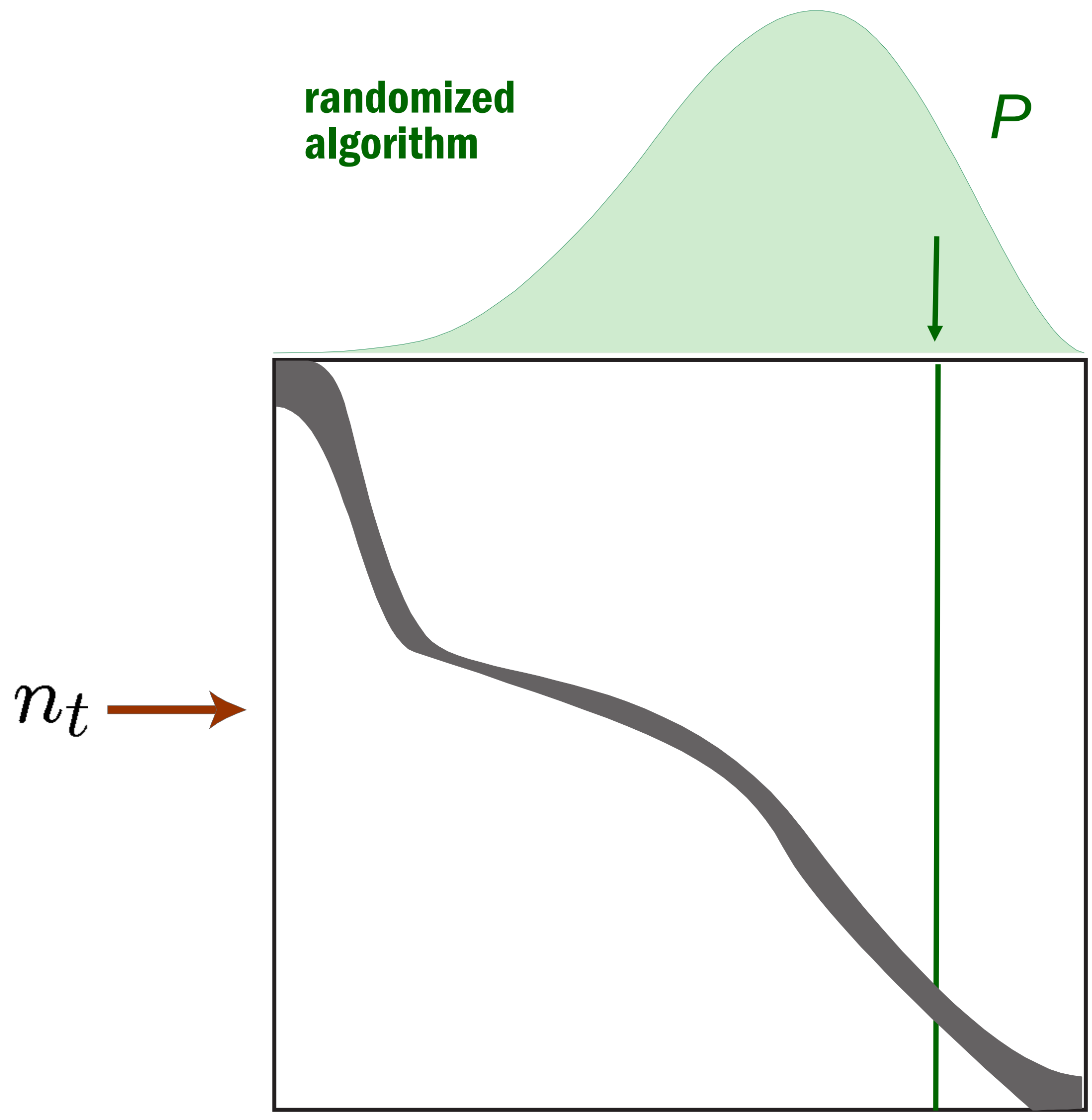

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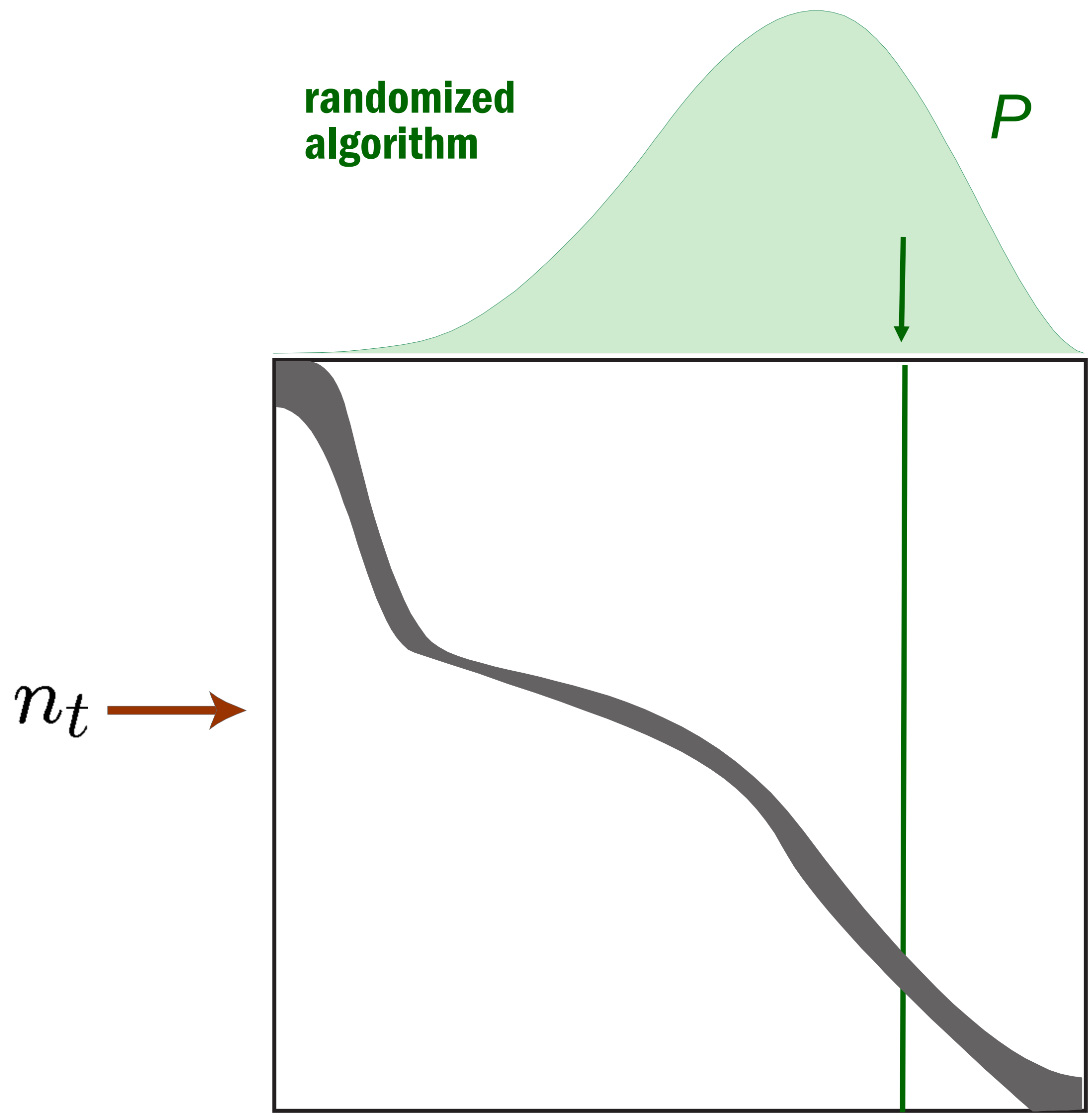

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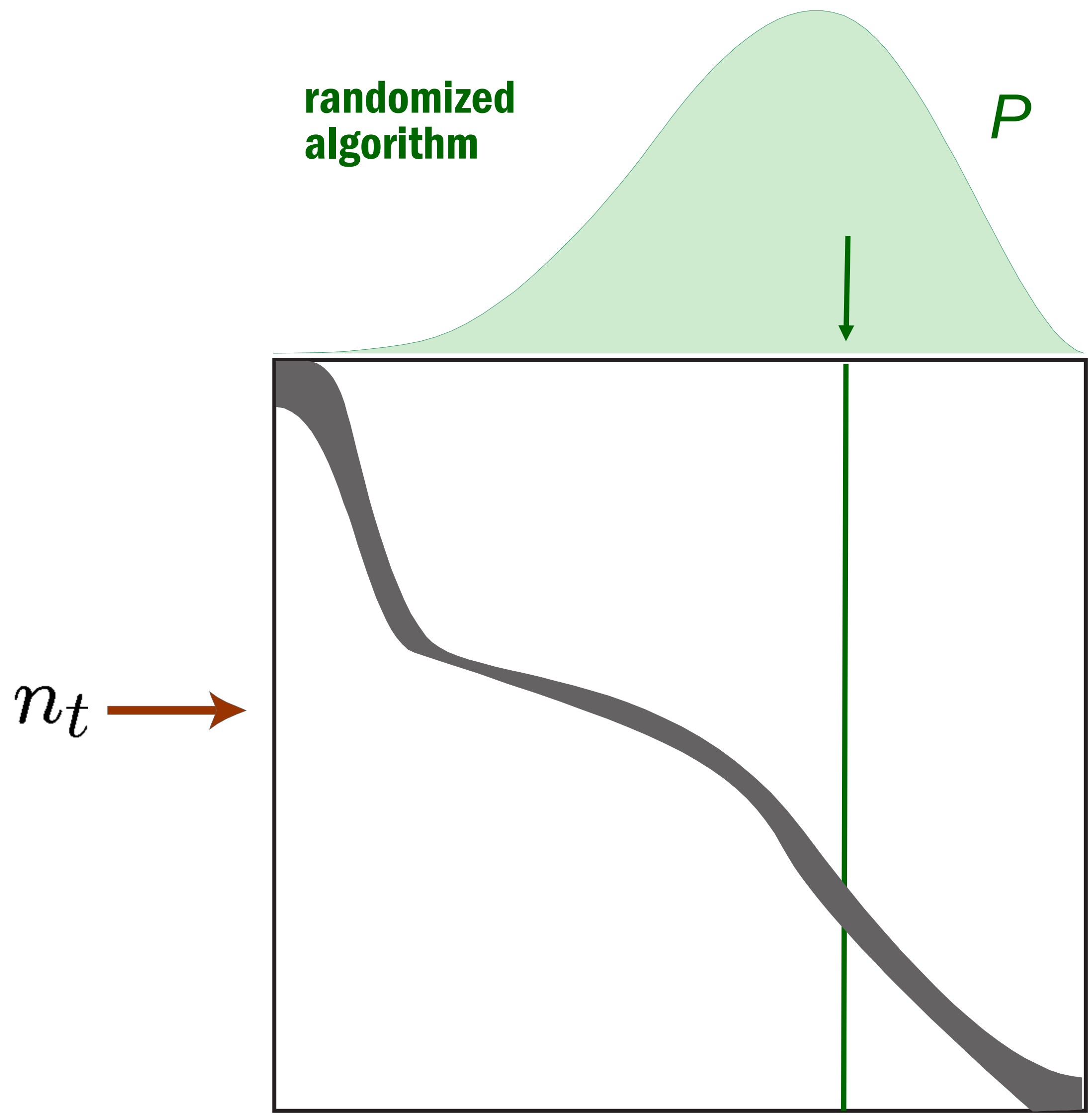

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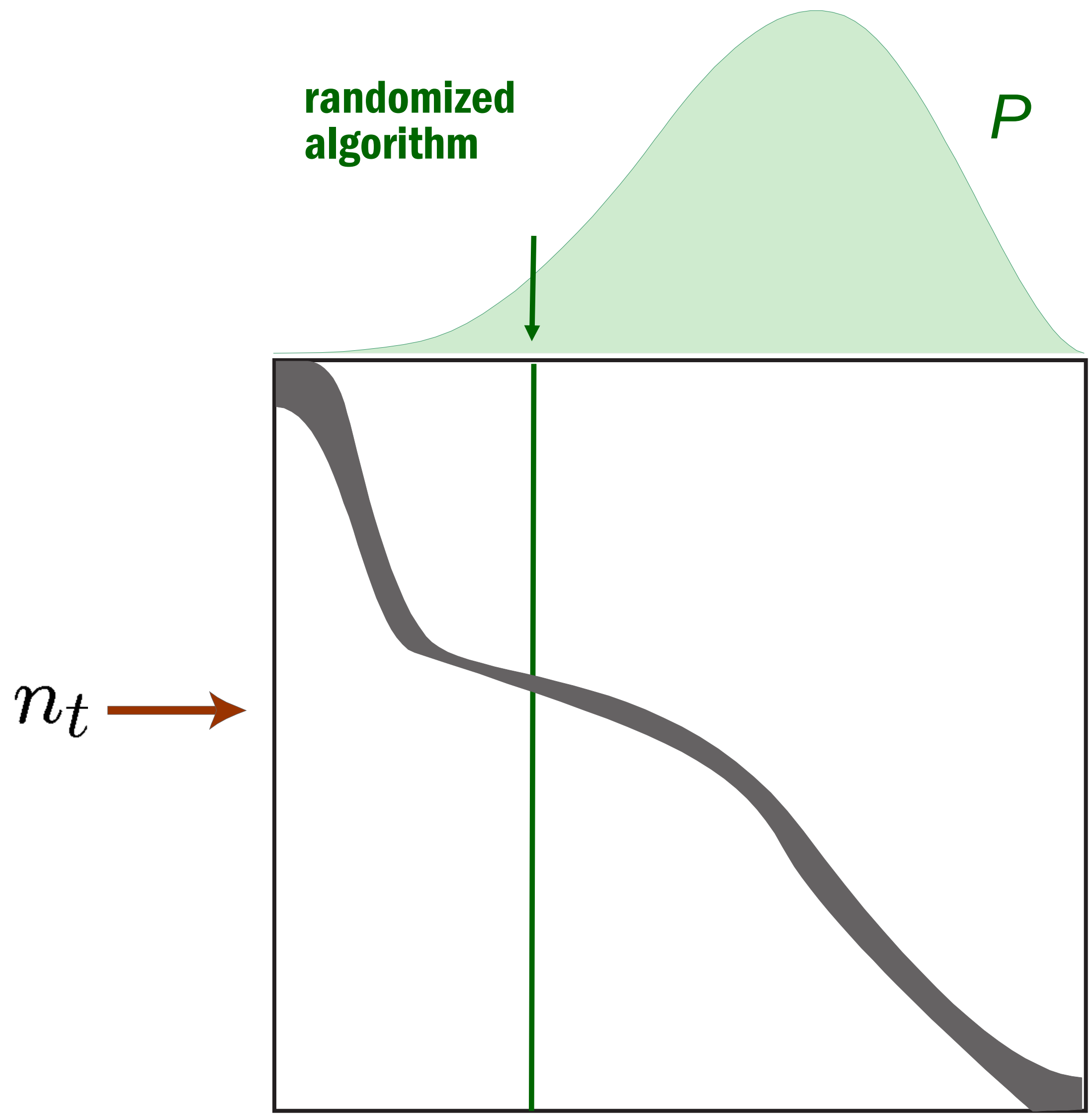

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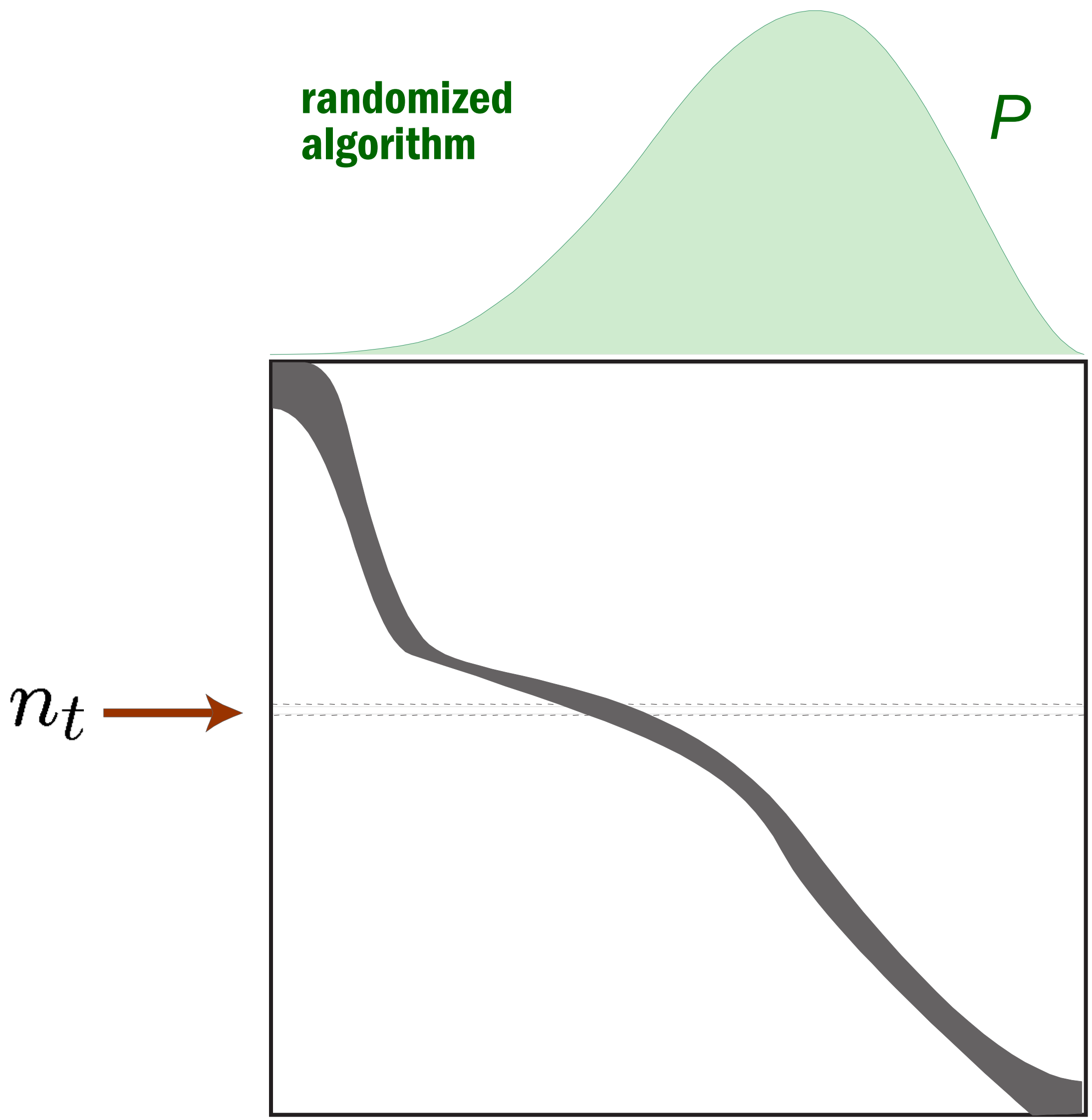

successful algorithm

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An algorithm is successful if, in all situations, it provides a correct answer.

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… a change of perspective:

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An algorithm is successful with probability p if, in all situations, its probability to provide a correct answer is at least p.

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Definition 2 (probabilistically successful algorithm)
An algorithm is successful with probability *p* if, in all situations, its probability to provide a correct answer is at least *p*.

this offers an extraordinary opportunity to satisfactorily solve “hopeless” problems

comments

- *P* exists in the algorithm

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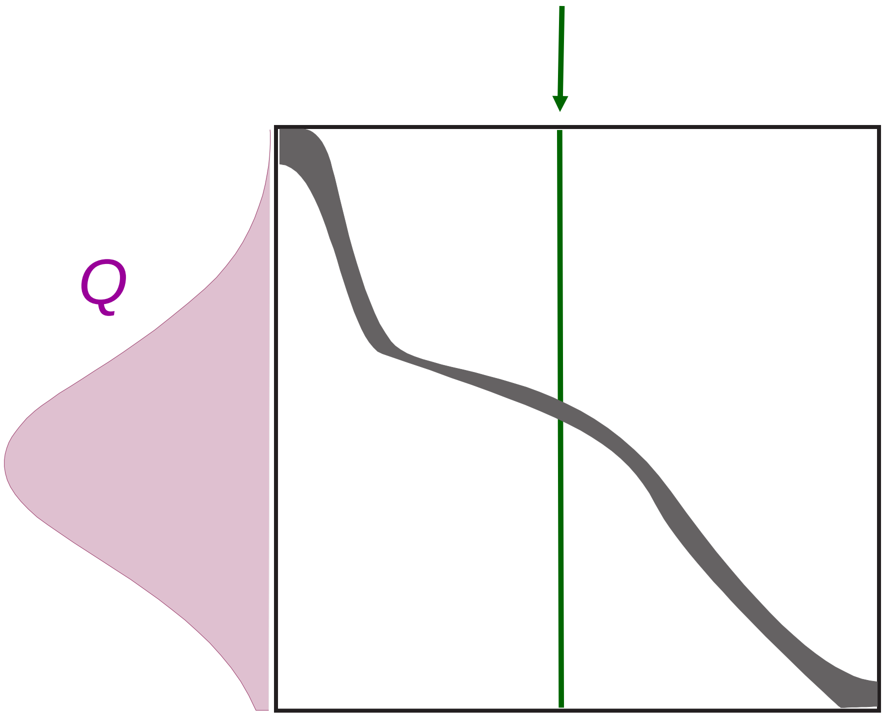

Bayesian perspective

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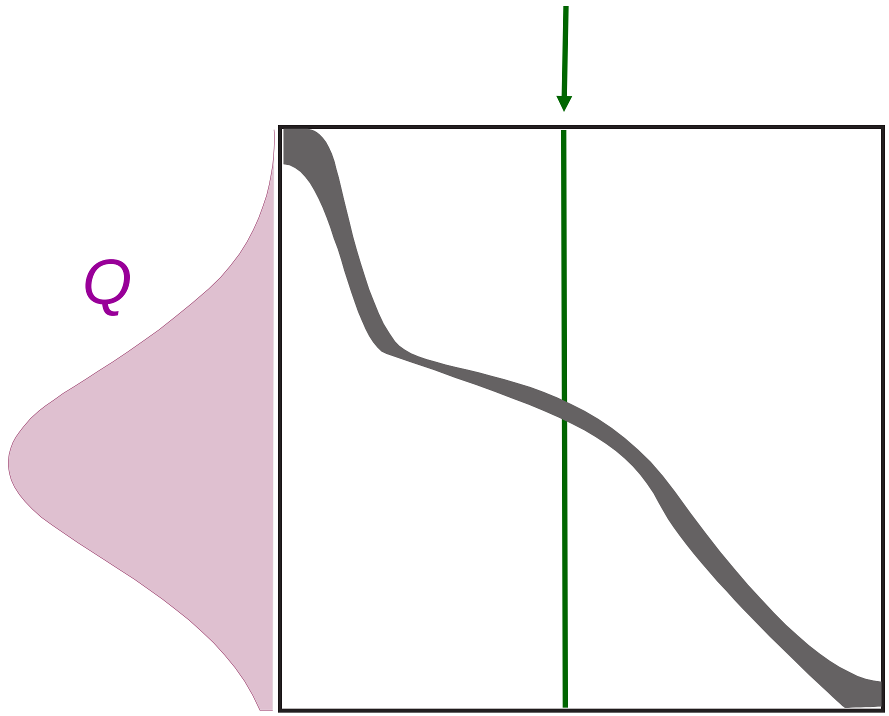

Bayesian perspective

(1) the result holds with high probability with respect to the situation of application

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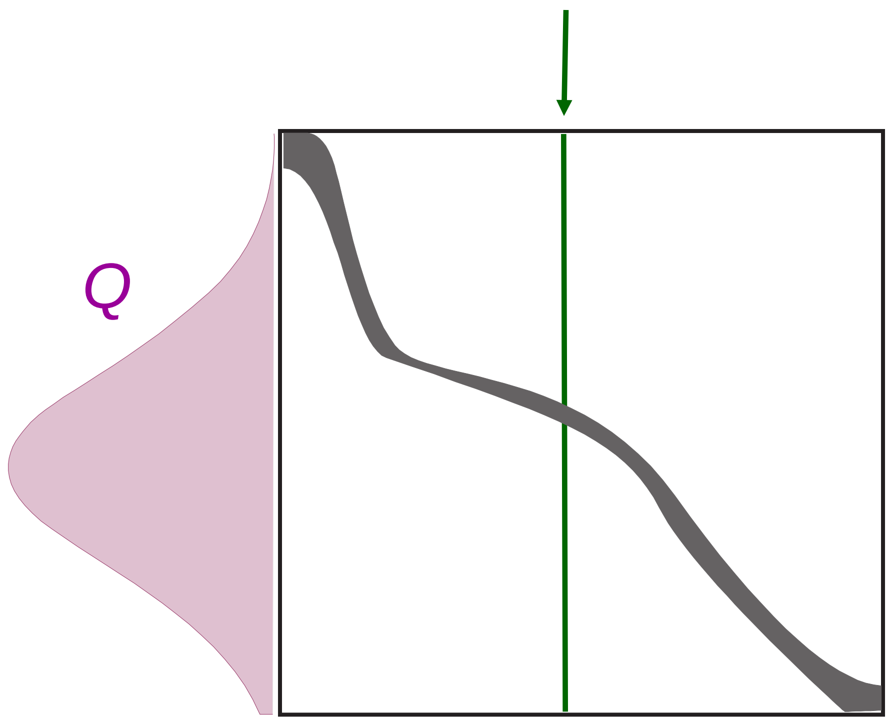

Bayesian perspective

(1) the result holds with high probability with respect to the situation of application

(2) *Q* describes reality → poor modeling

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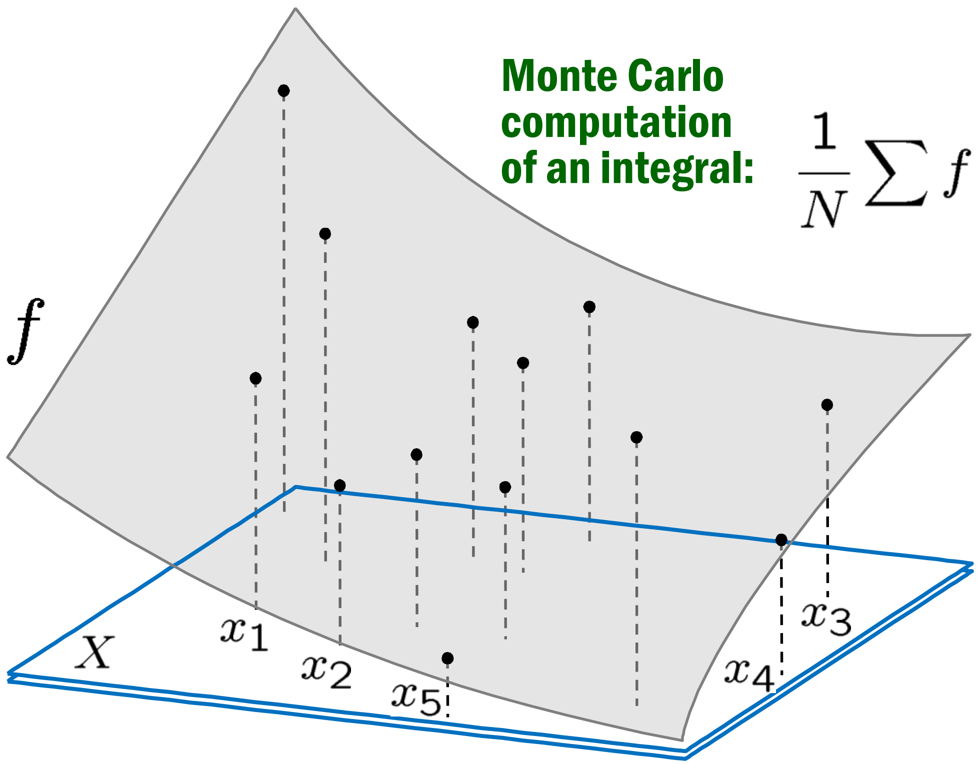

Monte Carlo computation of an integral: $\frac{1}{N}\sum f(x_i)$

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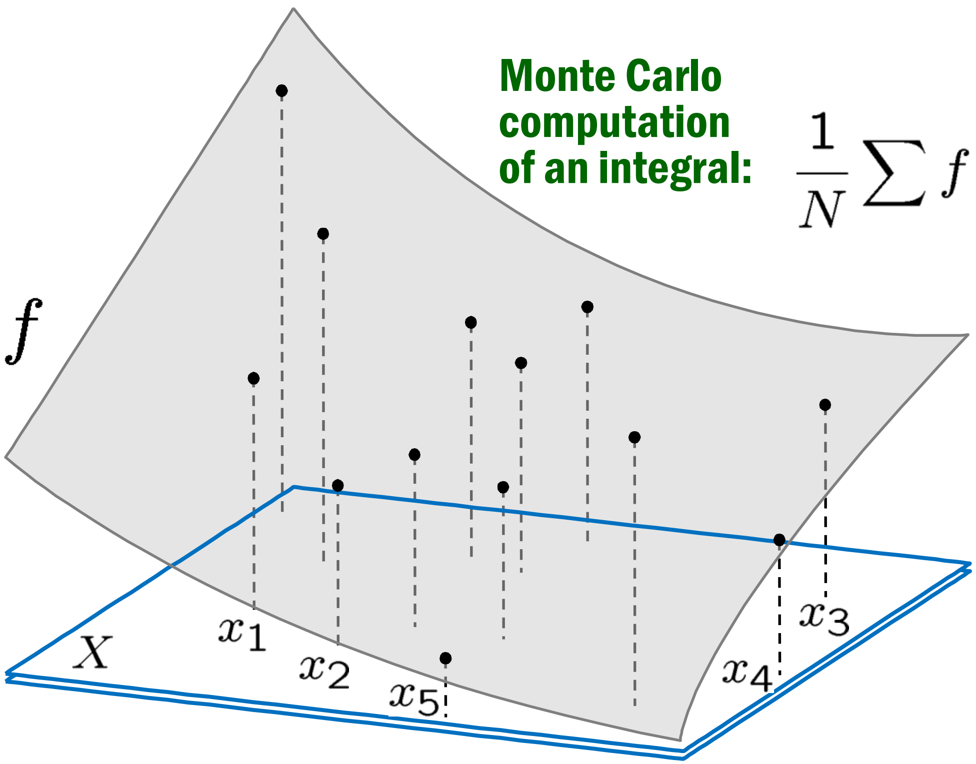

$$\left|\int f - \frac{1}{N}\sum f(x_i)\right| \leq 0.05$$

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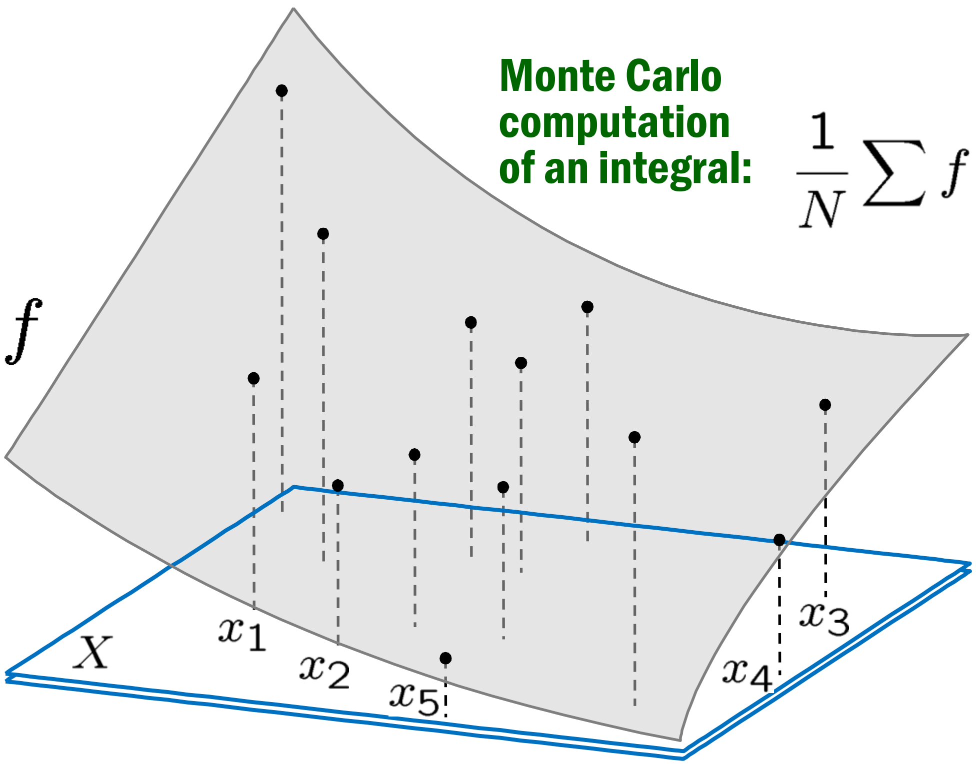

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$$\left|\int f - \frac{1}{N}\sum f(x_i)\right| \leq 0.05$$

$$\rightarrow \quad N = \frac{\ln[2/(1-p)]}{2(0.05)^2}$$

(Hoeffding's)

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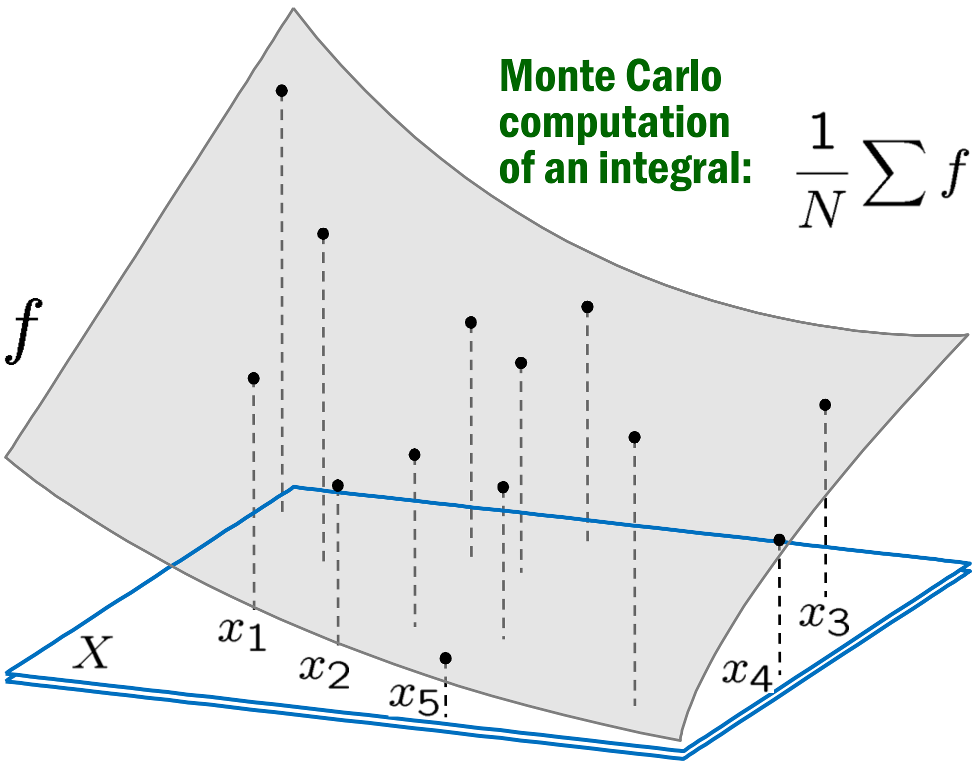

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$$p = 1 - 10^{-7} \quad \rightarrow \quad N = 3362$$

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comments

- P exists in the algorithm
- often the chance of failure can be made very small (concentration inequalities)
- amazingly powerful results from probability theory can be used to assess the probability of success

where are we?

where are we?

- Monte Carlo method (1949)
- computer science
  - sorting
  - counting
  - incremental geometric constructions
  - etc.
- optimization
  - simulating annealing
  - genetic methods
  - large-scale convex optimization
  - etc.

→ little in systems and control

where are we in systems and control?

- uncertain systems
  - aerospace
  - adaptive control
  - network control
  - etc.
- many contributors:

  T. Alamo, E.W. Bai, B.R. Barmish, T. Basar, G. Calafiore, A. Chaouki, F. Dabbene, B. De Shutter, L. El Ghaoui, M. Fu, Y. Fujisaki, S. Garatti, H. Ishii, S. Kanev, H. Kimura, C. Lagoa, S.P. Meyn, Y. Oishi, B. Polyak, M. Prandini, P. Shcherbakov, J. Spall, R.F. Stengel, M. Sznaier, V.B. Tadic, R. Tempo, B. Van Roy, M. Verhagen, M. Vidyasagar, K. Zhou

- R. Tempo, G. Calafiore, and F. Dabbene (2005). “Randomized algorithms for analysis and control of uncertain systems”. Springer-Verlag.

uncertain systems: design

robust min-max design (convex)

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robust min-max design (convex)

(non-convex $\longrightarrow$ Alamo, Camacho, Tempo)

a successful story of randomization: robust min-max convex design

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$$\min_{\theta} \left[\max_{\delta \in \Delta} \ell(\theta, \delta) \right]$$

a successful story of randomization: robust min-max convex design

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robust min-max design is hard!

Example: feedforward noise compensation

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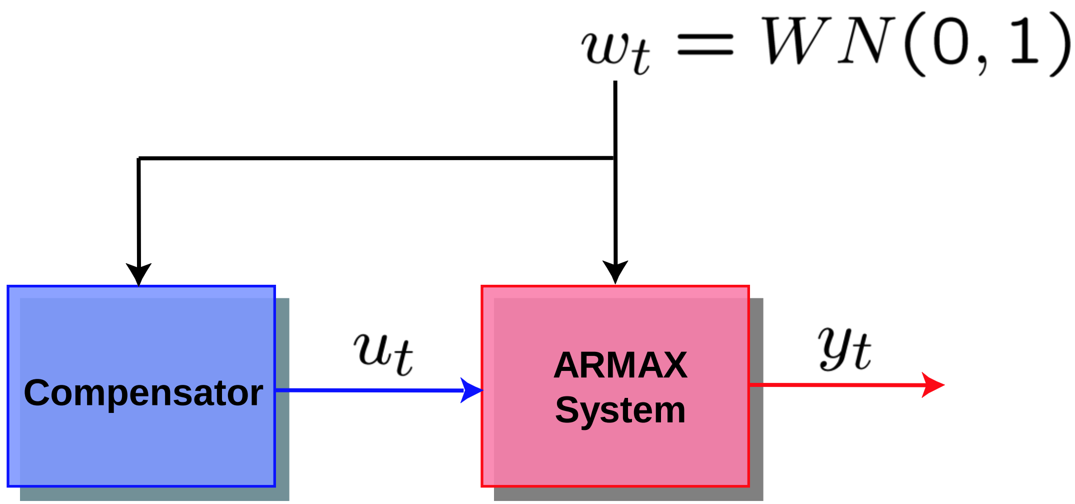
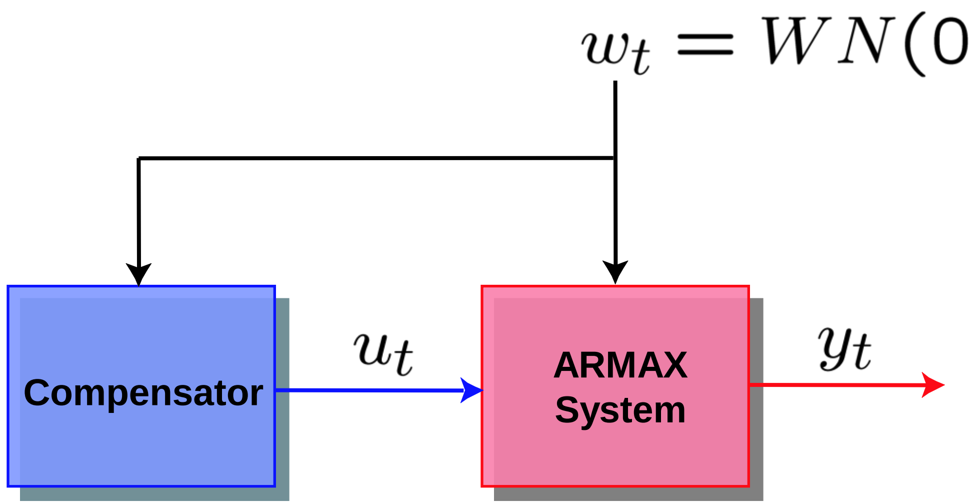
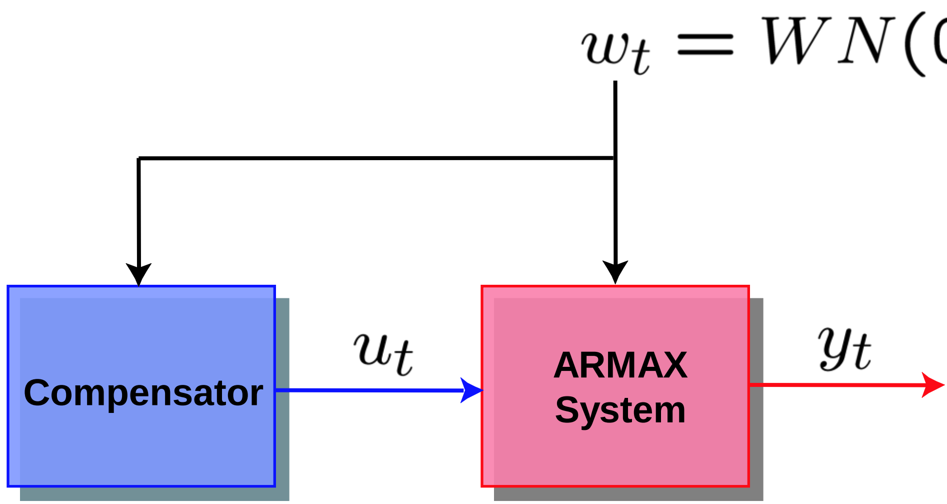


Example: feedforward noise compensation



Objective: reduce the effect of noise on y

Example: feedforward noise compensation



ARMAX System:

$$y_{t+1} = ay_t + bu_t + cw_t + dw_{t-1}$$

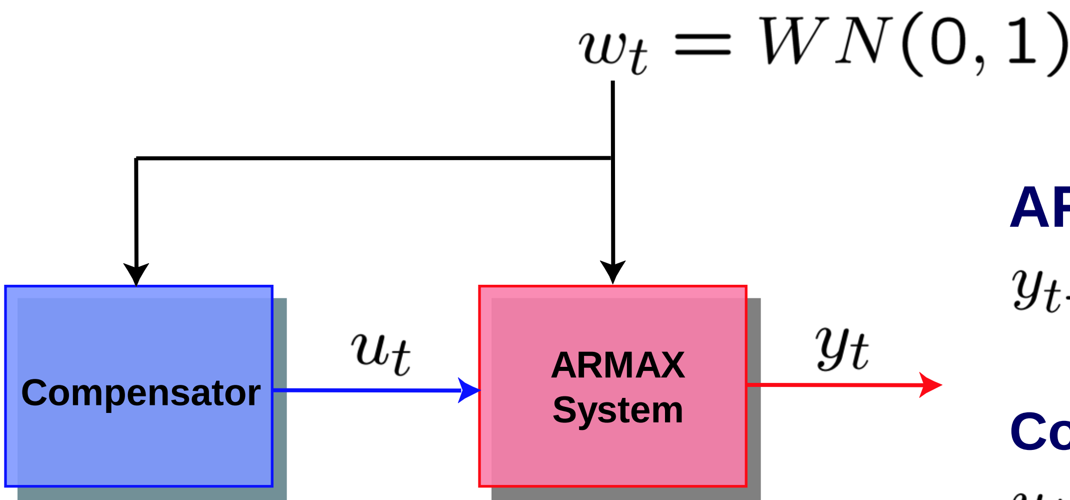
Compensator:

$$u_t = k_1 w_t + k_2 w_{t-1}$$

Goal:

$$\min var[y_t]$$

Example: feedforward noise compensation



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$$var[y_t] = \frac{(c + bk_1)^2 + (d + bk_2)^2 + 2a(c + bk_1)(d + bk_2)}{1 - a^2}$$

Example: feedforward noise compensation

system parameters unknown: $a, b, c, d \in \Delta$

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even a problem as simple as this is difficult for a generic Δ

other problems in robust control

- state-feedback stabilization
- H_∞ control
- H_2 control
- LPV control

$\vdots$

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  ⋮

… and systems theory

- model reduction
- prediction

  ⋮

- robust min-max design is hard!

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- what can we do?

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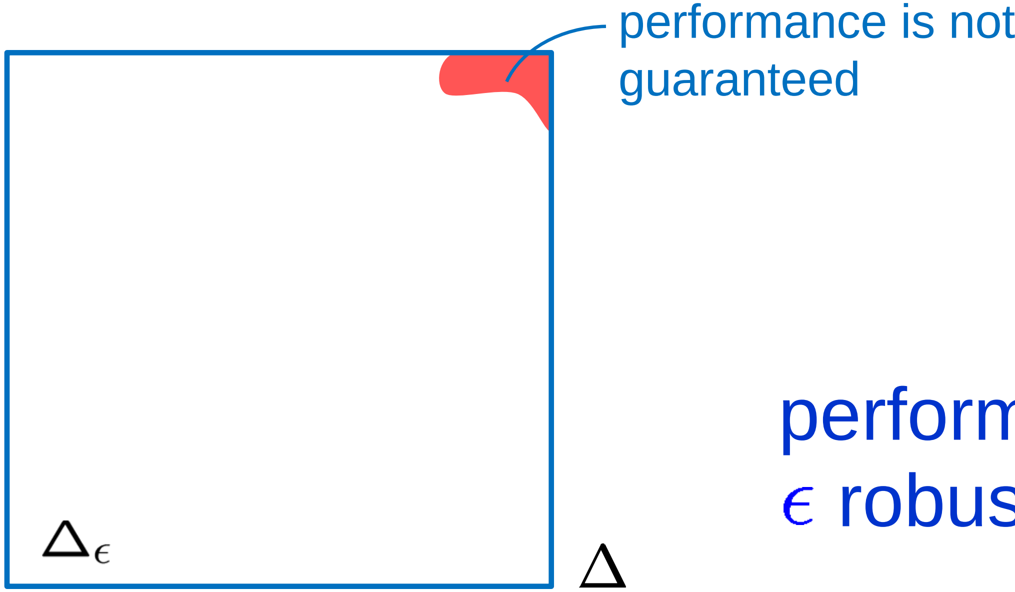
- what can we do?

- we need to accept a compromise

chance-constrained optimization

$$\min_{\theta} \left[\max_{\delta \in \Delta_\epsilon} \ell(\theta, \delta) \right]$$

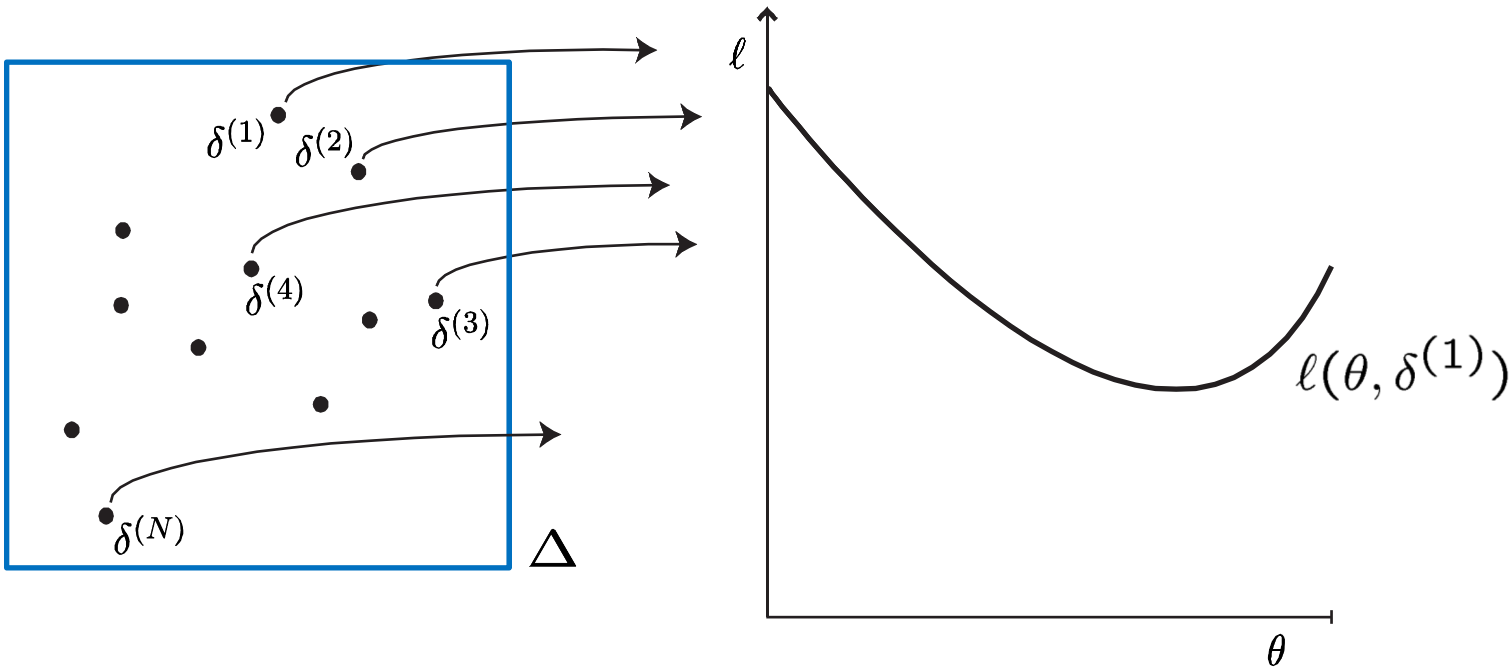
$$\frac{Vol(\Delta_\epsilon)}{Vol(\Delta)} \geq 1 - \epsilon$$



performance ℓ^* is ϵ robust

The “scenario” paradigm

(work done with G. Calafiore, S. Garatti)



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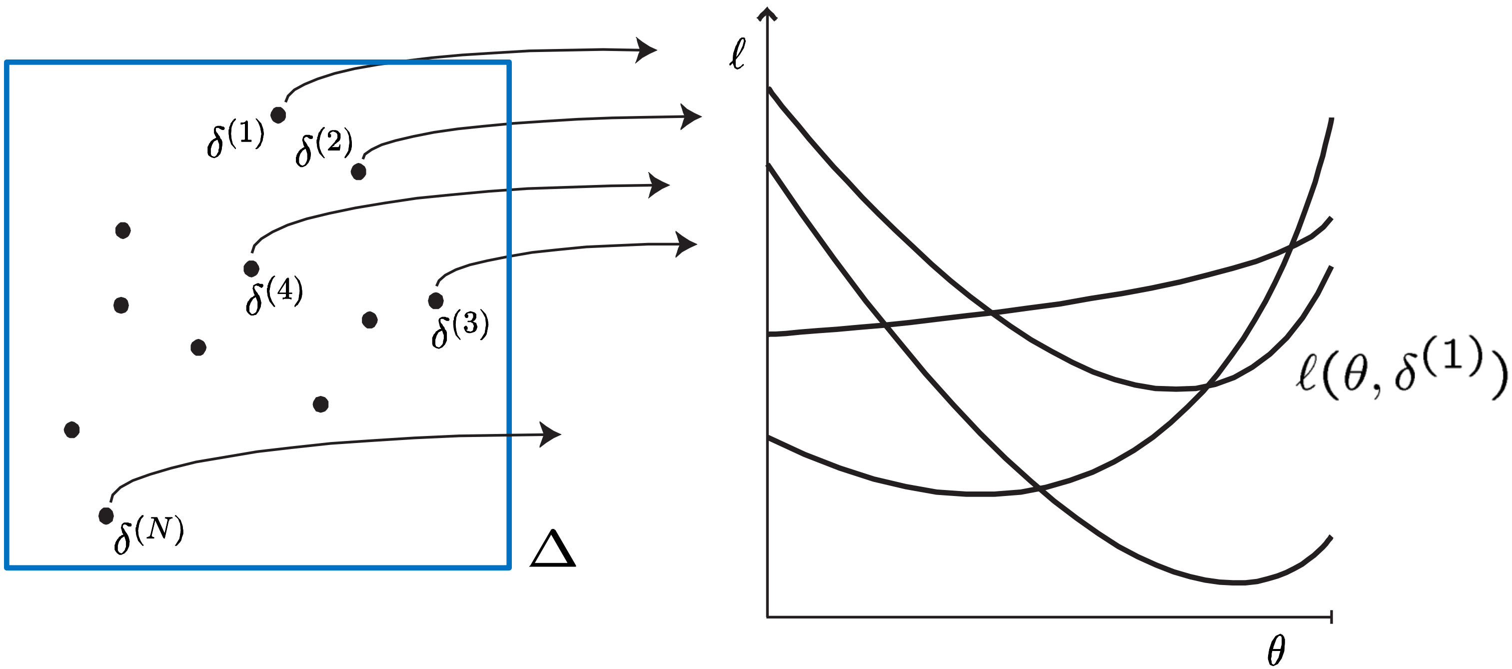

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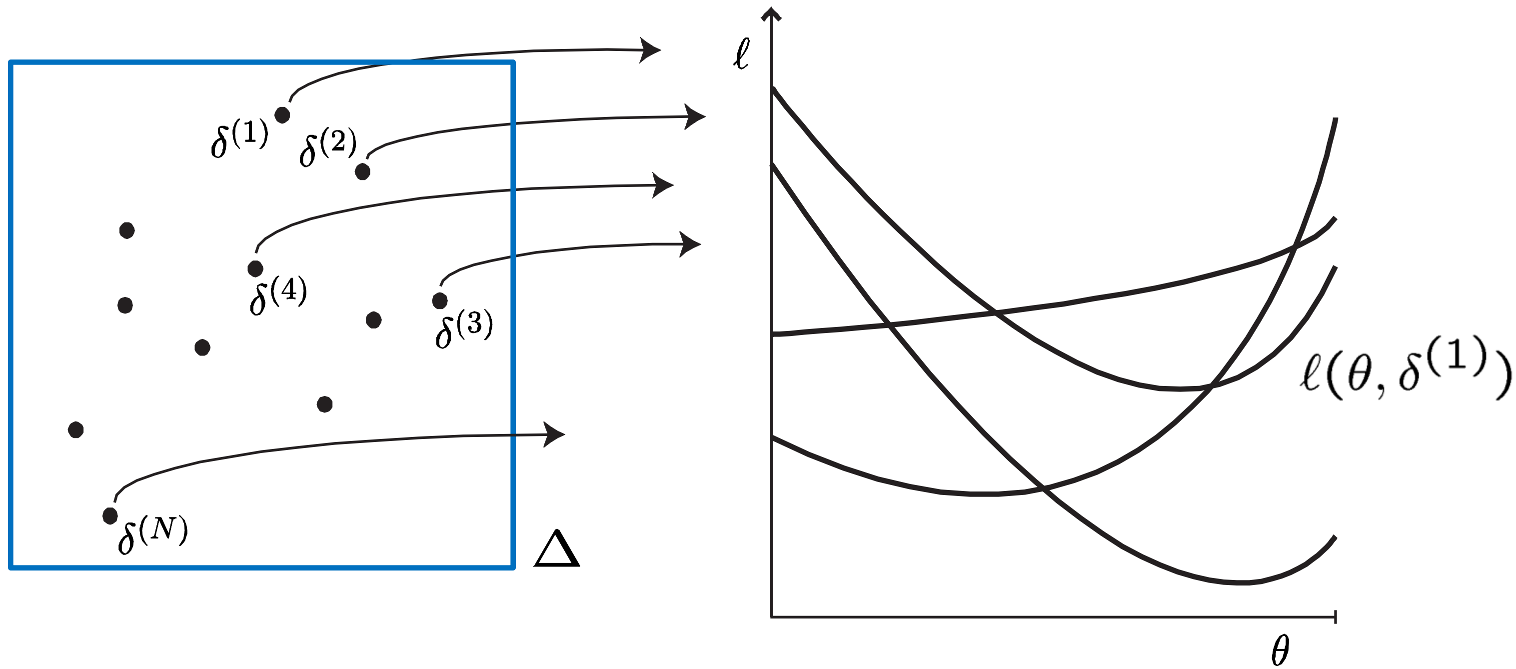

$$\min_{\theta} \left[\max_{i} \ell(\theta, \delta^{(i)}) \right]$$

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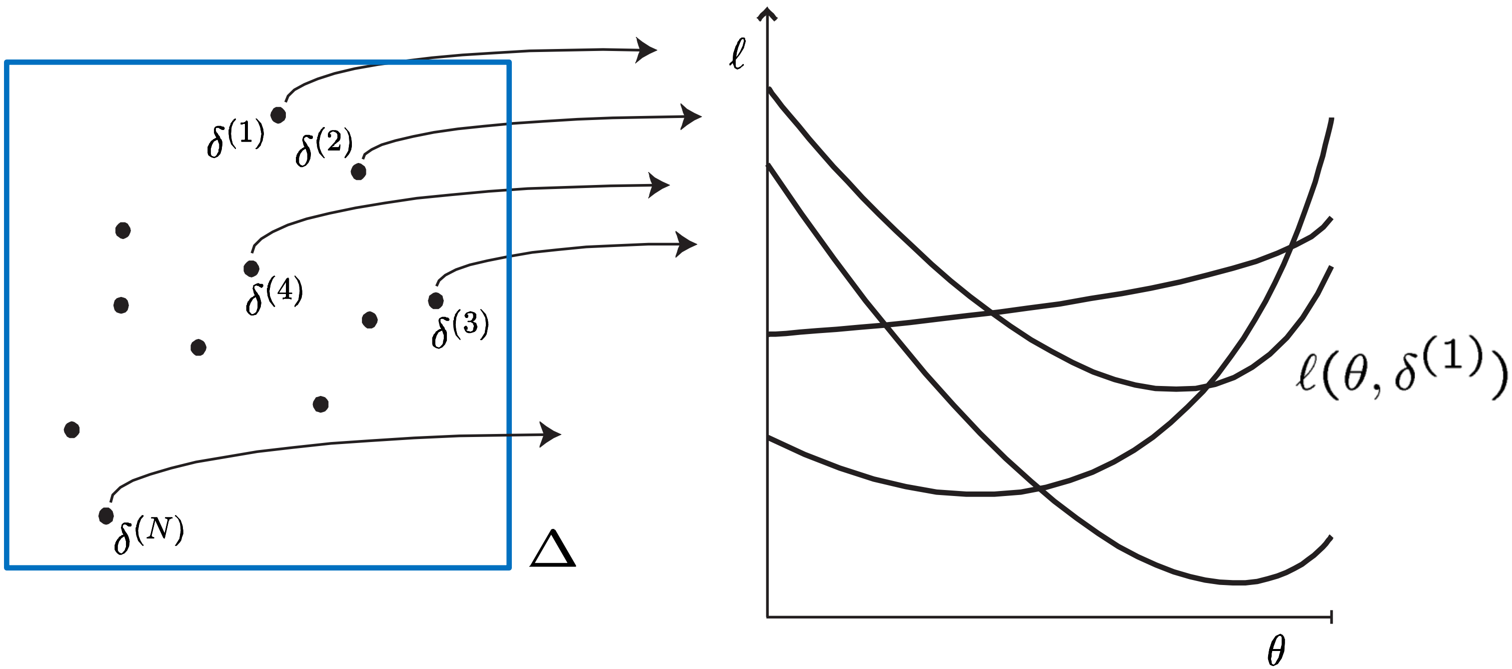

SP$_N$ = scenario program

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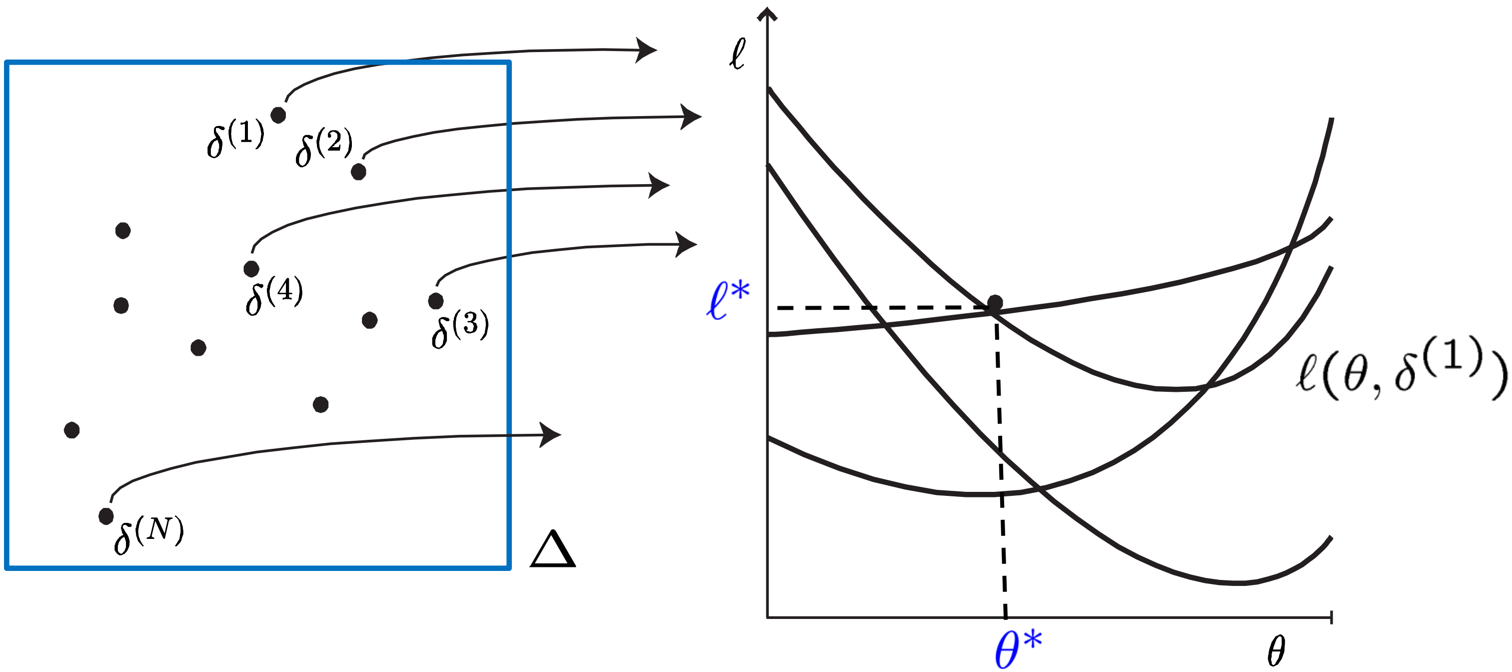

SP$_N$ = scenario program $$\min_{\theta} \left[\max_{i} \ell(\theta, \delta^{(i)}) \right]$$

- SP$_N$ is a standard finite convex optimization problem

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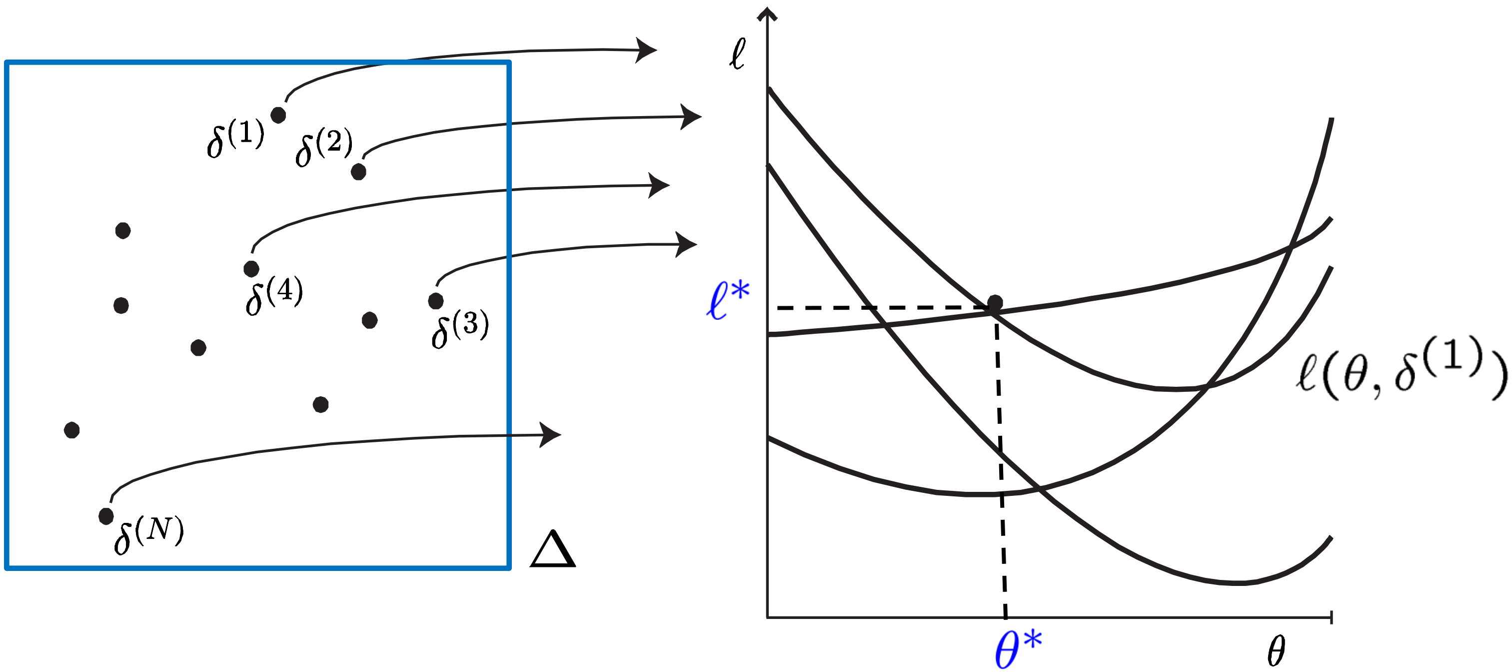

SPN = scenario program $\min_{\theta} \left[\max_{i} \ell(\theta, \delta^{(i)}) \right]$

- SPN is a standard finite convex optimization problem
- ℓ^* is superoptimal

Fundamental question:

How robust is ℓ^*?

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from the "visible" to the "invisible"

Theorem (with S. Garatti – G. Calafiore)

Fix $p \in (0,1)$ (probability of success)

If $N = \frac{2}{\epsilon}\left(\ln \frac{1}{1-p} + n_\theta\right)$,
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with probability of success p,
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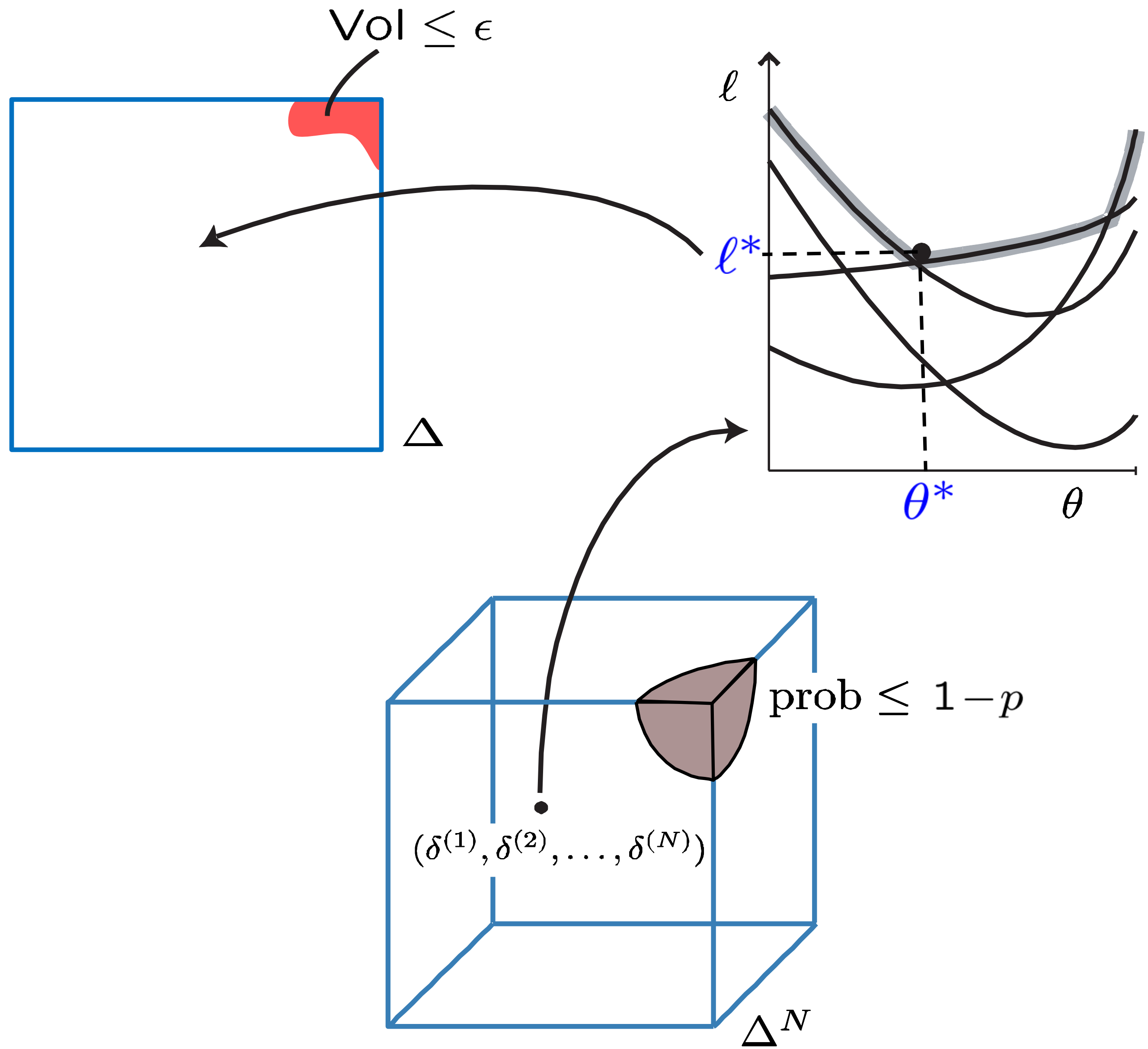
applicable to all convex problems!



Vol ≤ ϵ
Δ
ℓ
ℓ*
θ*
θ
prob ≤ 1−p
(δ(1), δ(2), . . . , δ(N))
ΔN

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$$\Delta = \{a, b, c, d \ : \ \begin{aligned} a &= 0.45 + 0.5 * (1 - \mathrm{e}^{-8\cdot 10^3(\sigma_1^2+\sigma_2^2)}), \\ b &= 1 + \sigma_2^2, \\ c &= 0.2 + (\sigma_2 + \sin(\sigma_2) + 0.1) \cdot \sin(2\pi\sigma_2), \\ d &= 0.5 + \sigma_1^2 \cos(\sigma_2), \\ \\ & (\sigma_1, \sigma_2) \in [-1/3, 1/3]^2\}. \end{aligned}$$

Example: feedforward noise compensation

$$\varepsilon = 0.01 \quad p = 1 - 10^{-7} \quad \Rightarrow \quad N = 2158$$

sample: $a_i, b_i, c_i, d_i \in \Delta, \quad i = 1, 2, \ldots, 2158;$

solve:

$$\min_{k_1,k_2} \left[\max_i \frac{(c_i + b_i k_1)^2 + (d_i + b_i k_2)^2 + 2a_i(c_i + b_i k_1)(d_i + b_i k_2)}{1 - a_i^2} \right]$$

$$\Rightarrow \quad k_1^* = -0.50, \quad k_2^* = -0.53, \quad \ell^* = 1.16$$

Example: feedforward noise compensation

$\ell^* = 1.16$ ⇨ **Output variance below 1.16 for all plants but a small fraction (1%)**

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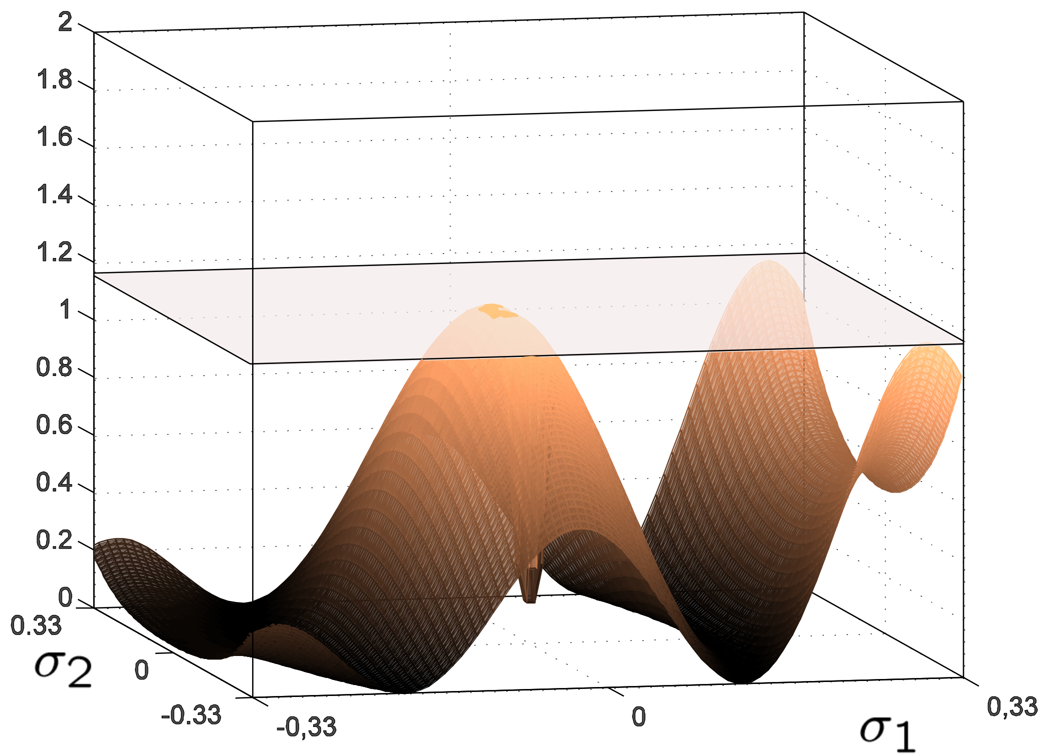

once more, a tough problem has turned into a solvable one through randomization, … provided we accept an ϵ risk

… in conclusion

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- randomization changes our perspective of problem solvability

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- the probability of success depends on an artificial *P* and can be assessed with extraordinarily powerful probabilistic tools

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each single problem has to be studied separately

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- the probability of success depends on an artificial *P* and can be assessed with extraordinarily powerful probabilistic tools

- it's just a paradigm; each single problem has to be studied separately

- can prove useful in many more problems in systems and control, especially at the boundary of control, communication and computation

THANK YOU

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